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ABSTRACT

This combination methods and resource book is written for preservice and inservice teachers of mathematics in the elementary school. It presents a program based on the research of the learning theories of Piaget, Brownell, Deines, Skemp, and Gagne and on successful classroom practices. The 14 chapters deal with: (1) "Mathematics in the Elementary School"; (2) "Assessment Procedures in Mathematics"; (3) "Planning for Teaching Mathematics"; (4) "Calculators and Computers"; (5) "Problem Solving"; (6) "Teaching the Foundations of Numeration"; (7) "Extending Understanding of Numbers and Numeration"; (8) "Informal Geometry"; (9) "Teaching Addition and Subtraction"; (10) "Teaching Multiplication and Division"; (11) "Fractional Numbers--Common Fractions"; (12) "Fractional Numbers--Decimal Fractions and Percent"; (13) "Measures and the Processes of Measuring"; and (14) "Tables, Graphs, and Probability." Appendices offer sources of information about math anxiety and equity; producers and distributors of mathematics courseware and learning aids; books for teachers; a guide to activities, problems and games; and an index. (PK)

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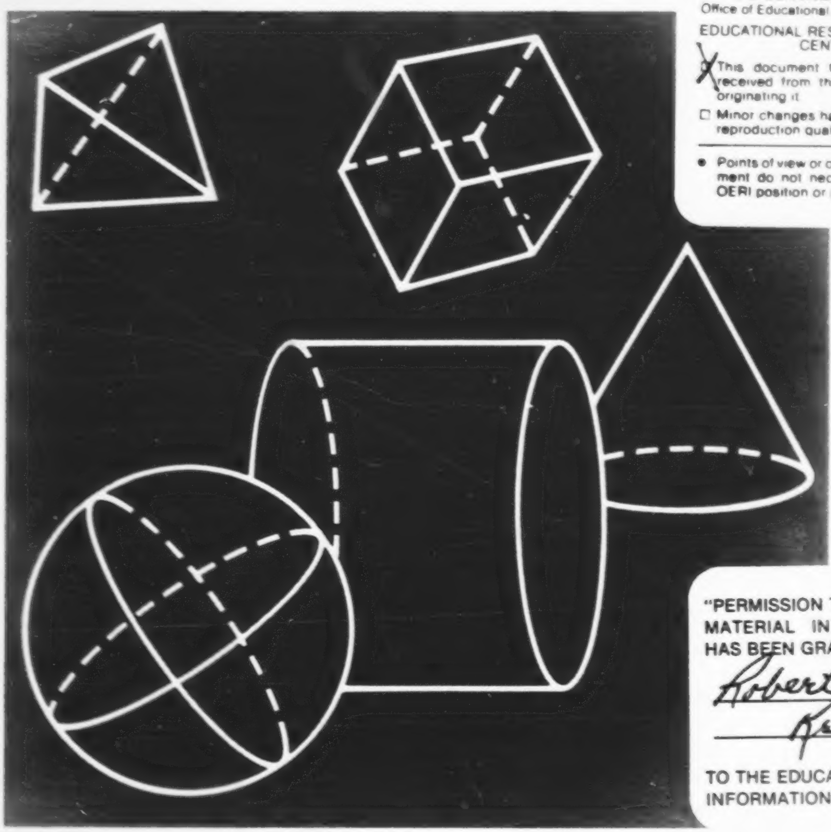
Guiding Children's Learning of Mathematics

Fifth Edition

Leonard M. Kennedy • Steve Tipps



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Guiding Children's Learning of Mathematics

Fifth Edition

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To Mary Kennedy and Rebecca Poplin

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Preface

This combination methods and resource book is written for preservice and inservice teachers of mathematics in the elementary school. It presents a program based on the research of the learning theories of Piaget, Brownell, Deines, Skemp, and Gagne, and on successful classroom practices. The book stresses the need for presenting mathematics through carefully sequenced activities in which the structure of mathematics is emphasized. Through such a program, children first develop an understanding of basic concepts and principles by engaging in activities with appropriate models of the mathematics they are learning. Upon achieving this understanding, children formulate the generalizations that make for more permanent learning and learn the computational and other skills they need to use mathematics effectively now and in the future.

The material in this text will help you present mathematics concepts in ways that lead children to see their relevance in a wide variety of situations. Numerous examples show you how a classroom can be organized to meet the varying needs of children; the roles of teacher-directed lessons, learning centers, individualized learning, and laboratories are clearly explained. Also discussed are the topics commonly found in today's elementary-school mathematics programs. Activities and materials for introducing each topic to children and developing their understanding of it are presented along with a wide variety of materials and procedures for dealing with each topic.

This book reflects current thinking about elementary-school mathematics and the ways in which it is presented to children. The book emphasizes careful teacher direction of lessons and activities with attention to sequencing children's work on each topic to help them attain an understanding of the topic. It retains the easy-to-read style, wide variety of problems and activities, and plentiful illustrations of earlier editions.

A major change in the book is information dealing with recent developments in computer applications in the classroom. Steve Tipps, the new coauthor, has used his experiences with computers in the classroom to contribute new activities dealing with Logo and commercial software for mathematics. He has also contributed additional problem-solving, calculator, and early-childhood activities throughout the book.

Changes In This Edition

The text has been extensively revised to reflect contemporary thinking about content, procedures, and uses of calculators and computers in the elementary-school mathematics classroom.

1. Chapter 1 discusses the basic topics included in a comprehensive program and the role of learning theories and classroom practices in the development of a theory of instruction.
2. Chapter 2 discusses 5 areas of readiness and ways to diagnose children's standing in relation to each factor. The need for a diagnostic-prescriptive approach to teaching is presented.
3. Chapter 3 discusses planning for presenting teacher-directed lessons, using learning centers, individualized learning plans, and laboratories. The importance of carefully planned activities is stressed. Also discussed are children with special needs, mathematics equity, and mathematics anxiety.
4. The chapter on computers and calculators introduces the idea that both of these tools should be integrated throughout elementary mathematics. Calculators and computers are no longer limited to computation and practice activities. The calculator is an aid for problem solving and for exploring concepts in mathematics. Improved educational software materials that promote reasoning allow children to explore mathematics and enhance their understanding of mathematics with the computer. A new evaluation form is included as a guide for selecting software. The Logo computer language is also presented as a computer use that emphasizes thinking.
5. Chapter 5 stresses problem solving. Twelve strategies are presented with example problems for each strategy. The solutions to problems have been moved to the end of the chapter so that the instructor can use the problems in class. A section about classroom problem solving is new in this chapter. Polya's 4-step prob-

lem-solving process is adapted for elementary students, and suggestions are given to improve children's work with word problems and estimation.

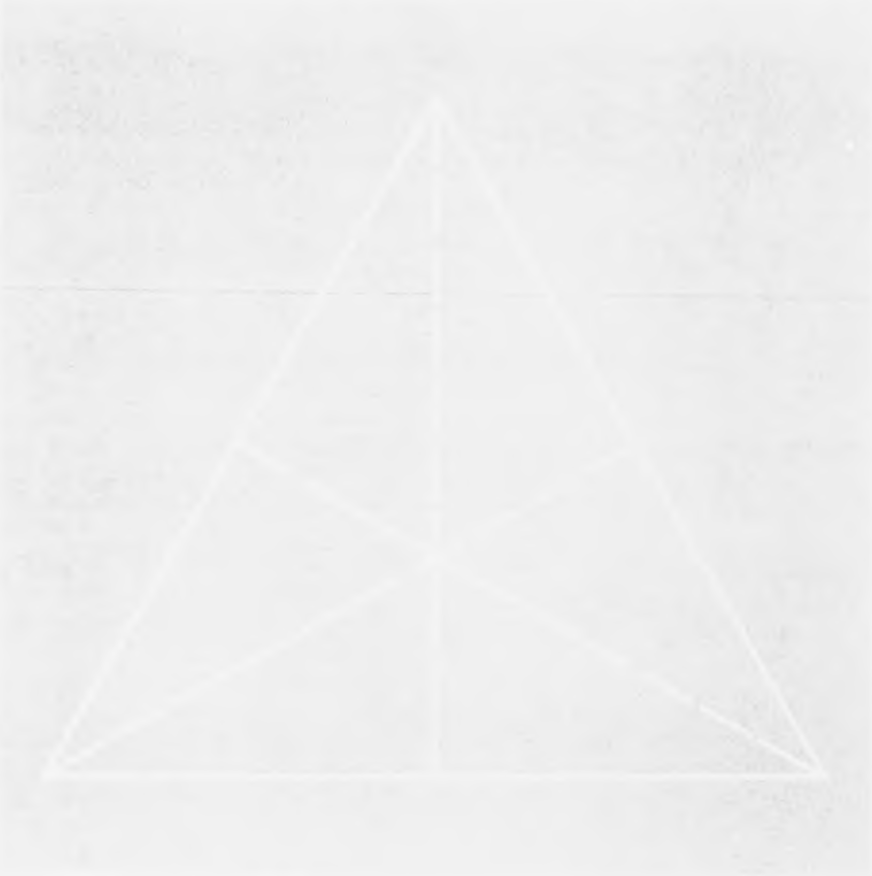
6. Chapters 6 and 7 deal with whole numbers. Each chapter stresses the need for manipulative experiences before attempting pictorial, representational, and abstract work with numbers.
7. The chapter on geometry (8) has been moved ahead of those dealing with numbers and operations to emphasize the importance of geometry in a comprehensive mathematics program. Many activities with concrete materials enabling children to explore plane and space concepts in geometry are discussed. In addition, a new section illustrates how children can use Logo to learn about shapes and numbers. A classroom vignette shows how a teacher may intervene to help students generalize the mathematics they explore with Logo.
8. Operations with whole numbers are discussed in Chapters 9 and 10. Fractional numbers expressed as common and decimal fractions and percent are the subject of Chapters 11 and 12. A rationale for teaching common fractions is included along with materials and procedures for teaching the 4 operations with both common and decimal fractions. Discussion of the role of the computer and calculator relating to these operations is included.
9. Chapter 13 contains information about measurement, emphasizing the metric system. Activities for children's experimentation with measurement concepts in nonstructured situations, using both nonstandard and standard units of measure, are presented along with sections that deal with instruction about money and telling time.
10. The final chapter presents new information about tables, graphs, and probability experiments. Many new activities, both teacher directed and computer assisted, are included.
11. Many new classroom activities using manipulative materials, computers, calculators, and estimation and mental arithmetic have been added. A new, quick guide to help make selection of activities, problems, and games easy is located at the end of the book. Specific software materials that promote mathematical thinking or knowledge have been suggested in many chapters. Short Logo procedures have also been included in several chapters.

The authors are indebted to James Babb, Minot State College; Douglas K. Brumbaugh, University of Central Florida; Robert C. Clark, Florida State University; Louise Mayock, Chestnut Hill College; K. Allen Neufeld, University of Alberta, Edmonton; Carole J. Reesink, Bemidji State University; Melvin E. Rexroat, Idaho State University; and Sharon L. Young, formerly

of Louisiana State University, Baton Rouge. We believe that most of the book's reviewers will notice some way in which their views have been reflected in this fifth edition. Naturally we accept full responsibility for any issues or topics they or anyone else may find inadequately covered. We hope that readers will find much that is valuable for helping them guide children's learning of mathematics.

Guiding Children's Learning of Mathematics

Part 1



Foundations

Chapter

1

Mathematics in the Elementary School

In Chapter 1, you will read about

1. problem solving, calculator and computer use, estimation and mental computation as essential skills in today's mathematics programs
2. seven topics included in a comprehensive mathematics curriculum
3. the background information needed by an elementary mathematics teacher, including basic operations and their properties
4. the importance of manipulatives in teaching mathematics
5. four types of mathematical behavior that are needed for success in an increasingly complex world
6. a teaching-learning sequence suggested from research about learning and successful classroom practice

Every teacher faces the challenge of preparing children to live in an ever-changing world. Today's elementary school children will reach adulthood after the beginning of the twenty-first century. The rapidity with which changes occur in today's world makes it impossible to predict what life will be like in even a few years, let alone when children become adults. Therefore teachers must help prepare children for a life of continuous learning in order to cope with unknown changes to come.

For teachers of mathematics in the elementary school, the challenge means helping children acquire a background of understanding and skills in mathematics that will allow them to face the next century with confidence. The focus of mathematics in today's schools has shifted from an emphasis on memorizing number facts and paper-and-pencil computing to developing a broad range of problem-solving skills, because these skills will give children a better preparation for the future.

MATHEMATICS SKILLS

Problem-solving Skills

A major emphasis of mathematics today is the development of problem-solving skills. And an important concern of every teacher is that children acquire mathematical understandings and skills that will permit them to apply what they know in new and unexpected situations. When confronted with a problem, each student should be able to engage in meaningful, though not necessarily sequential, mental or physical activities that lead to the problem's solution. In a classroom, a teacher must provide problem-solving situations that are appropriate for particular groups of children. Whenever possible, these situations are drawn from children's in- and out-of-school experiences and supplement or replace the problem situations presented in their mathematics textbooks.

Calculator Skills

The development of calculator skills is another emphasis in today's classroom. Calculators have become pervasive in today's society because of the ease with which they permit computation to be completed. It is recommended in this book that by the end of the sixth grade, children have skill in using calculators for all appropriate areas of computation and problem solving. Calculators should be regularly used by children in the classroom, even for tests, and also for assignments completed at home. This does not mean that children need not learn the meaning of the computational operations they perform, with or without machines. It means that instead of spending many tedious hours practicing pencil-and-paper calculations involving addition, subtraction, multiplication, and division, children should spend classroom time learning the meaning of various computational processes. Then they are ready for learning how to distinguish between situations where using calculators is appropriate and those where another process should be used to complete actual computations. Calculators free children from time-consuming practice exercises, which lead to little mathematical understanding, to engage in more productive activities that deal with other areas of mathematics including both real and hypothetical problems.

Estimation and Mental Arithmetic Skills

Estimation and mental arithmetic take on new importance in programs where problem solving and calculators are emphasized. Both skills must be regularly and consistently developed so that children's number sense, understanding of mathematical processes, and problem-solving skills are enhanced. Estimation is used to determine the reasonableness of an answer and to determine the possible range of potential answers in a given situation. The California State Department of Education lists these estimation skills as examples of important ones that children should learn:

a. Comparison: 48 + 38. Because both numbers are less than 50, the total is less than 100.

b. Clustering: 42 + 56 + 63 + 37. By observing that two of the numbers are close to 40 and two are close to 60, the sum can be estimated as (2 × 40) and (2 × 60).

c. Rounding fractions: 9/10 + 4/9. Since 9/10 is about 1 and 4/9 is about 1/2, the sum is approximately $1\frac{1}{2}$.

d. Reasonableness of context: The height of a building cannot reasonably be 5 inches or 5 feet; the height of a person cannot be 5 inches or 50 feet.

e. Compatible numbers: 235 ÷ 12. Since 235 is close to 240, a multiple of 12, the quotient can be estimated as 240 ÷ 12 = 20.[1]

Mental arithmetic provides children with computing skills they can use in place of calculator or paper-and-pencil procedures. An example of a mental procedure for whole numbers is one in which the child uses a compensation process. While adding 56 and 27, the child thinks "56 plus 30 equals 86," then subtracts 3 from 86 to determine the answer—83. An example for multiplication illustrates use of the distributive property. When multiplying 42 and 50, the child thinks "40 times 50 is 2000; 2 times 50 is 100; 2000 plus 100 is 2100."

---

[1] California State Department of Education, *Mathematics Framework for California Public Schools Kindergarten Through Grade Twelve* (Sacramento: The Department, 1985), p. 4.

Some children develop and use a wide range of mental arithmetic processes independently of any instruction they receive. To other children it never occurs that computation can be done in such ways. A teacher's responsibility is to help all children learn to compute answers mentally as well as to use a calculator and paper and pencil.

Computer Skills

Computers are another essential part of the modern mathematics curriculum. Using a computer to organize, analyze, and display information is an important skill in mathematics and other subjects. While the computer was originally thought of as a teaching machine for directing learners to correct answers, a newer idea holds that the computer is a learning machine with which the learner dynamically interacts to create and solve problems. The computer provides opportunities for students to interact with mathematics in some very different ways than are possible with paper and pencil.

Only in recent years have computers become accessible to large numbers of children in schools and homes. This rapid change has caught many schools and teachers off guard and unsure of ways to proceed. The instructional possibilities include practicing, tutoring, simulating, gaming, demonstrating, and testing.[2] Software for each possibility is available for almost every topic in mathematics. As the demand for software has increased, its quality has constantly improved. While a comprehensive guide to using computers for mathematics is beyond the scope of this book, the authors will suggest ways in which you might use a computer to help you make informed choices about using the machines for mathematics instruction. A computer gives you a valuable tool for individualizing instructions by fitting mathematics to the interest and skill levels of all children.

---

[2] Larry Hatfield, "Toward Comprehensive Instructional Computing in Mathematics," in *Computers in Mathematics Education, 1984 Yearbook* (Reston, Va.: National Council of Teachers of Mathematics, 1984), pp. 1–9.

MATHEMATICS TOPICS INCLUDED IN THIS BOOK

Before children can use calculators, computers, and mental arithmetic to help themselves solve problems and use estimation to judge the reasonableness of answers, they must have a good understanding of mathematics topics and processes. It is clear that mathematics instruction that emphasizes paper-and-pencil computation, along with a few measurement and geometric concepts and skills will not be sufficient for America's children. A much broader range of understandings and skills is needed by students and must be provided. The following mathematics topics are typical of those found in contemporary classrooms throughout the country:

Numbers, Numerals, and Operations
Geometry
Measurement
Patterns and Functions
Probability and Statistics
Tables and Graphs
Number Theory

Each of these topics is discussed in one or more chapters of this book.

Numbers, Numerals, and Operations

Most time spent on mathematics in the elementary school is devoted to numbers and operations. By the time children have completed the sixth grade, they have had experiences with whole numbers and fractional numbers expressed as decimal and common fractions and percents. Some will have had experiences with integers expressed as negatives of the nonzero whole numbers. Children have learned that whole numbers are used to count objects in collections, or sets; that fractional numbers are used when part of a unit or part of a set is considered; and that integers are used for expressing temperature, time, distance, and altitudes, and for other situations in which both direction and magnitude must be indicated. Children must have a clear understanding of each of these types of numbers and of the Hindu-Arabic, or base-ten, numeration system, which provides the words and symbols for expressing numbers. Otherwise their abilities to count, compute, and measure, and to use mathematics to solve problems will be severely limited.

The basic number operations are addition and multiplication and their inverses, subtraction and division. It is not enough that children learn procedures for computing each operation, whether with calculator, paper and pencil, or mentally; a clear understanding of real-world situations that give rise to an operation must also be developed. Initially children learn the meaning of an operation by engaging in activities with things that they manipulate as they count, group and ungroup, and compare sets of objects. Gradually, as their understanding of an operation matures, they learn one or more paper-and-pencil procedures, or algorithms, for computing answers for that particular operation. For example, a common procedure for completing the subtraction in the margin is the decomposition algorithm. When this algorithm is used, it is necessary to regroup a ten and rename it as 10 ones, giving a total of 13 ones and leaving 7 tens. Now the subtraction can be completed by subtracting 6 from 13, 2 from 7, and 4 from 5 and recording the answer—157. (There are other algorithms for computing this subtraction; one of them is discussed in Chapter 9.) Once an operation and an algorithm are understood, children should also learn to use calculators and any appropriate mental arithmetic processes to complete calculations for that operation.

$$\begin{array}{r} 583 \\ -\ 426 \\ \hline \end{array}$$

$$\begin{array}{r} 5\overset{7}{\cancel{8}}{}^{1}3 \\ -\ 426 \\ \hline \end{array}$$

$$\begin{array}{r} 5\overset{7}{\cancel{8}}{}^{1}3 \\ -\ 426 \\ \hline 157 \end{array}$$

There are properties that apply to certain of the basic operations which children can learn to use in meaningful ways. For example, they can change the order of addends when they add without affecting the sum: 3 + 4 yields the same answer as 4 + 3. This demonstrates the commutative property for addition. The commutative property also applies to multiplication but not to subtraction and division.

The associative property also applies to both addition and multiplication but not to subtraction and division. In the set of whole numbers, $(3 + 4) + 6 = 3 + (4 + 6)$. In the first part of the equation, the parentheses indicate that 3 and 4 are to be added first, and then 7 and 6 are to be added to make 13. In the second part, the parentheses indicate that 4 and 6 should be added first, and then 3 and 10 are added to make 13. In either half of the equation, the sum is 13. An example for multiplication is $(2 \times 5) \times 6 = 2 \times (5 \times 6)$.

$$\begin{array}{r} 36 \\ \times\ \ 4 \\ \hline \end{array}$$

$$\begin{array}{r} 36 \\ \times\ \ 4 \\ \hline 24 \\ 120 \\ \hline 144 \end{array}$$

The distributive property applies to multiplication over addition. It makes it possible to complete a multiplication algorithm in stages when numbers like 4 and 36 are multiplied. To complete the algorithm in the margin, 36 is thought of as 3 tens and 6 ones. The product of 4 and 6 is recorded first; then the product of 4 and 30 is recorded. The sum of these partial products is then recorded. The distributive property is also used when performing division. The division of 36 by 3 can be explained as $36 \div 3 = (30 + 6) \div 3 = (30 \div 3) + (6 \div 3) = 10 + 2 = 12$.

Children also learn that there are identity elements for operations performed on numbers. An identity element is a number that when operated on with another number results in the other number. Zero is the identity for addition. In the set of whole numbers, $3 + 0 = 3$. In multiplication the identity is one, that is, $1 \times 3 = 3$.

Number sentences are used to record and communicate mathematical ideas. As children study mathematics, they learn that number sentences are a form of shorthand used to make statements about number situations. One important problem-solving skill is to analyze a problem situation and write a number sentence for it. Once a number sentence is determined, it can be rewritten in algorithm form so the necessary computation can be performed.

Geometry

Geometry has a prominent role in elementary school mathematics. Through activities both inside and outside of the classroom, children develop an intuitive understanding of basic geometric concepts.

We regularly use geometric shapes in both human-made and natural objects. In human-made objects, such shapes are often a necessary part of the object's structure. Geometric shapes are used in many ways for aesthetic purposes; their uses in art from all periods of history are widely recognized. In school the study of geometry begins with an examination of objects so that children can develop an understanding of two- and three-dimensional figures and the vocabulary associated with them. Later, carefully selected models of cubes, prisms, and pyramids give opportunities for classifying figures according to the shapes of their faces and the number of edges and vertices. As children mature, they refine the language they use to describe each form. Other geometric concepts explored in the elementary school include similarity, congruence, and symmetry; transformations, including translations (slides), rotations (turns), and reflections (flips); and coordinate geometry.

Measurement

Children's in- and out-of-school activities bring them into frequent contact with both discrete and continuous materials. Discrete materials are things that are counted, such as marbles and oranges. Continuous materials are things that cannot be counted, such as the weight of a package of meat and the length of a piece of ribbon. Continuous materials are measured rather than counted. The types of measures treated in this book are linear, perimeter, area, volume (both capacity and cubic units), weight, and temperature. Also included are activities dealing with time and money.

When the Metric Conversion Act was signed into law by President Gerald Ford in 1975, it was believed that the United States would make a transition from the English (customary) system of measures to the metric system. It was anticipated that the changeover would be completed in a relatively few years. Such has not been the case, however, and both systems are used today. As long as children live in a society where both systems are used, they will need to learn about and use units for the two systems. Units from the metric system are used as examples for activities in this book. Teachers can easily substitute customary units for metric units in each of the activities to give children experiences with them.

Patterns and Functions

A study of patterns includes activities dealing with visual patterns, such as one created with colored beads on a cord or colored blocks covering a surface in some systematic way; and with number patterns, such as those in the sequence of whole numbers. Working with geometric and numerical patterns helps children develop skill in inductive reasoning and in organizing and understanding observations of their physical and mathematical worlds.

The functions that children deal with are associated with points on a coordinate plane and with ordered pairs identified by the points. The study of functions also includes activities dealing with coordinate graphs.

Probability and Statistics

The preface of the 1981 yearbook of the National Council of Teachers of Mathematics, *Teaching Statistics and Probability*, contains five reasons why statistics and probability are appropriate topics for school mathematics:

> They provide meaningful applications of mathematics at all levels.
>
> They provide methods for dealing with uncertainty.
>
> They give us some understanding of the statistical arguments, good and bad, with which we are continually bombarded.
>
> They help consumers distinguish sound use of statistical procedures from unsound or deceptive uses; and
>
> They are inherently interesting, exciting, and motivating topics for most students.[3]

Elementary school children are not likely to face situations in which an understanding of probability and statistics is crucial, but situations requiring some use of these topics do arise frequently and are of importance and interest to them. Work with statistics can begin as early as kindergarten, where children can make surveys to determine their favorite television shows or video games or the kinds of pets each one has. As the years pass, children's skills in handling data meaningfully should be nurtured as they collect, organize, display, and interpret information from a variety of sources. Simple experiments give children opportunities to develop an intuitive understanding of certain probability concepts and procedures. A strong emphasis should always be placed on the application of knowledge about statistics and probability in problem-solving situations.

---

[3] Albert P. Schulte, ed., *Teaching Statistics and Probability, 1981 Yearbook* (Reston, Va.: National Council of Teachers of Mathematics, 1981), p. ix.

An additional benefit of activities with statistics and probability is that they frequently provide opportunities to work with arithmetic skills such as addition, subtraction, multiplication, and division. Much of the work with these topics can be done through independent individual and small-group investigations.

Tables and Graphs

Tables and graphs are used to organize data so that trends and patterns can be determined and problems involving data can be solved. Among the graphs children learn to make and interpret are object, picture, bar, circle, and line graphs. Object and picture graphs can be used by kindergarten and first-grade children to organize information about themselves, such as their favorite ice cream flavors and their birthday months. Older children use picture, bar, circle, and line graphs to organize and display their own data and to make more sophisticated interpretations and predictions using data from tables and graphs prepared by themselves and others.

Number Theory

The study of number theory introduces children to odd and even numbers, prime and composite numbers, and concepts such as least common multiple (LCM) and greatest common factor (GCF). There are several reasons for including number theory in a program. Children who are familiar with the concept of least common multiple can use LCMs to determine the lowest common denominators when adding and subtracting unlike common fractions. Knowledge of greatest common factor is helpful when simplifying common fractions (reducing to lowest terms).

Another reason for including number theory is that it is interesting to many children. A properly taught unit gives a program a motivating factor that might otherwise be missing. The study of number theory also provides many opportunities for children to practice mathematical skills, such as addition or multiplication involving the basic facts for these operations.

FOUNDATIONS FOR EFFECTIVE TEACHING

A mathematics curriculum as comprehensive as the one described in the previous section presents a challenge to every teacher who wishes to develop and offer a program that helps every child acquire a sound background of mathematical knowledge, skills, and understandings. What background do you, who will ultimately be responsible for the mathematics instruction of a group of children, need in order to meet this challenge? For one thing, you must know the mathematics you will teach. If you do not know all the mathematics involved in the topics needed for a broad curriculum, you should plan now to overcome your weaknesses before you begin working with children. You can do this by studying mathematics content books for teachers or by enrolling in appropriate courses in a community college or a four-year institution.

You also need to know how research into learning, research into teaching mathematics, and classroom practices provide the bases for a sound mathematics program. Since all children are not alike, you need knowledge of ways to assess children's understanding of mathematics and skills in performing operations and their attitudes toward the subject. You need to know how to organize a classroom for effective instruction. You should be familiar with some of the issues related to equity in mathematics for all students. Mathematics anxiety inhibits the learning of mathematics for many persons; you need to know some of its causes and ways to lessen its influence. Naturally you need knowledge of materials and procedures for teaching specific topics.

The remainder of this chapter focuses on how research into learning, teaching, and successful classroom practices influence ways in which mathematics should be organized and taught. Subsequent chapters of Part 1 deal with assessment of children's knowledge and understanding of mathematics and their attitudes toward the subject; with classroom organization, equity and mathematics anxiety issues; and with special students, calculators and computers, and problem solving. Part 2 is devoted to materials and procedures for teaching specific mathematics topics.

Three Classroom Vignettes

The three situations that follow illustrate lessons by a first-grade, a third-grade, and a fifth-grade teacher. Each lesson deals with subtraction of whole numbers.

The first-grade teacher introduced the concept of "take-away" subtraction by having children engage in activities dealing with materials from their "math boxes."

Each box contained a collection of related objects, such as small plastic airplanes, cars, or snakes. The children were directed to count a set of five airplanes (cars, snakes). After the teacher verified that each child had counted the correct number of objects, she instructed each one to remove two airplanes (cars, snakes). Again she verified the work. Finally she asked the children to count the number of objects that remained in each set. Sets of other sizes were manipulated and discussed in the same way until the children were able to describe what "take-away" subtraction is.

To help his children grasp the meaning of the decomposition algorithm for subtracting the number 295 from 362, a third-grade teacher had them use file cards bundled in groups of 100 and groups of 10, along with loose cards. He presented the following situation: "There were 362 greeting cards in a rack on Monday. By week's end 295 cards had been sold. How many cards remained in the rack at the end of the week?" The teacher discussed the nature of the situation and possible means of solution with the children. Among the ways to determine the answer was use of the cards. The children showed 362 with three bundles of 100 cards, six bundles of 10 cards, and 2 loose cards. The children acknowledged that they could not remove 5 loose cards from the collection as now constituted. Previous activities with bundled cards and other place-value devices gave them the knowledge they needed in order to regroup one bundle of 10 cards as 10 loose cards to go with the 2 cards they already had. Now 5 loose cards could be removed from the 12 cards. In a similar way, a bundle of 100 cards was regrouped as ten bundles of 10 cards, giving fifteen bundles of 10 cards and leaving two bundles of 100 cards. Nine bundles of 10 cards and two bundles of 100 cards could now be removed. The answer—67—was determined by counting the cards that remained. Other problem situations were presented, and answers were determined in a similar manner.

The fifth-grade teacher wanted to expand her children's mental arithmetic repertoire. She had determined that they were ready to learn how to subtract a number like 2945 from 6003 without paper and pencil or a calculator. She used this dialogue with her class:
Teacher: I want to show you how you can subtract 2945 from 6003 without using pencil and paper or a calculator. Put a thumb up if you can tell me how many tens there are in 6003. (Some children answered that there weren't any. The response the teacher wanted and got from some children was that there are 600.) If we regroup one of the tens as 10 ones, how many tens are left?
Children: There are 599 tens.
T: Let's see how what we have just done can be used to subtract 2945 from 6003. Think: "There are 599 tens and 13 ones." What is 5 from 13?
C: 8.
T: What is 4 from 9?
C: 5.
T: What is 9 from 9?
C: 0.
T: What is 2 from 5?
C: 3.
T: Now raise a thumb if you can tell me the answer to 6003 minus 2945.
Child: 3058.
This process was repeated a number of times until the children could do the mental subtraction with similar pairs of numbers.

The teachers who prepared these lessons were aware of what researchers into learning and the teaching of mathematics have said about how mathematics should be presented to children. They also knew about some of the materials and procedures used by successful teachers in classrooms like their own. Knowledge of research and successful classroom practices helped them develop lessons that they knew would be effective.

Research Into Learning

Research into learning provides information you can use to develop a sound mathematical program. During the late part of the nineteenth century, the mental discipline theory had a great influence on mathematics teaching. According to this theory, the mind is like a muscle and benefits from exercise as muscles do. Mathematics was used to give the mind much of its exercise, and lengthy computations that were largely devoid of meaning were contained in textbooks. Instruction emphasized ways to carefully copy and compute answers.

Early in the present century, Edward Thorndike's stimulus-response theory supplanted the mental discipline theory. His theory was based on the belief that learning occurs when a bond or connection is established between a stimulus and an appropriate response. Mathematics lessons consisted primarily of presenting many number combinations so that children could establish the bond between each combination and its response (answer). Drill was heavily emphasized.

Later in this century William Brownell, Zoltan Dienes, Jean Piaget, Richard Skemp, and Robert Gagne contributed findings that support a mathematics program that emphasizes the development of an understanding of the mathematics being learned. In the 1930s Brownell introduced his meaning theory, which is based on the belief that children must understand what they are learning if learning is to be permanent. This theory supports children's use of many objects to manipulate so that they can understand the meanings of new concepts and skills.[4]

Zoltan Dienes's research led to the idea of the "multiple embodiment" of a concept. His work convinced him that various representations of a concept are needed in order for children to fully understand it. Multiple embodiment of place value occurs when bundles of cards, beansticks, multibase blocks, Unifix cubes, and a classroom abacus are used.[5] (These and other place-value devices and their uses are discussed in Part 2.)

Jean Piaget and Richard Skemp concluded that all individuals pass through stages as they mature. Piaget identified four stages through which he believed all individuals pass.[6] Not until children reach the fourth stage, called formal operations, are they fully capable of analyzing all factors in situations, forming abstract hypotheses, and testing these hypotheses against reality. Transition into this stage begins for some children around age twelve and continues into adulthood. During earlier stages children do not have the mental maturity to grasp fully those mathematical concepts that are presented by words and symbols alone. They need experiences with many concrete objects in order to represent abstract ideas and the operations involving those ideas.

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[4] For an interesting review of research in mathematics during the first 50 years of this century, see William A. Brownell, "The Revolution in Arithmetic," *Arithmetic Teacher*, XXXIV, No. 2 (October 1986), pp. 38–42. The article is a reprint of the original article that appeared in the first issue of the *Arithmetic Teacher* in February 1954.

[5] Zoltan P. Dienes, *Building Up Mathematics* (London: Hutchison Education, 1960).

Skemp[7] does not define stages in the same way Piaget did. Skemp separates learning into two levels. At the first level, manipulation of objects in the in- and out-of-school environments provides the learner with activities that form the basis for further learning and the internalization of ideas. Later these earlier experiences form the basis for learning at the abstract, or second, level. Skemp also believes that learners must form a conceptual structure, or schema, of their experiences before learning can be useful. This means that children must understand the structure of mathematics before they can use prior knowledge for further learning and apply their knowledge effectively to the solution of problems. Skemp advocates a physical environment that gives children opportunities to interact with objects during the early stages of learning a concept. They also need time to reflect and a chance to act on knowledge so they can organize their thoughts and internalize their learnings.

Most of Gagne's work deals with investigations into what he calls the hierarchy of steps through which a learner must pass in order to complete a task.[8] He has concluded that learning is improved when the subtasks needed to arrive at a specific task are first identified and ordered so they can be taught in a sequence until the target task has been accomplished. Figure 1-1 shows one sequence that might be used for instruction about subtraction with whole numbers. The target task or goal is the ability to subtract a pair of numbers of any size. The subordinate tasks leading to the target task are listed in a sequence, beginning with the simplest one at the bottom of the flow chart. According to Gagne, learning is facilitated by assuring that a student understands each of the subtasks in sequence until the target task is accomplished. The accomplishment of target tasks in elementary school

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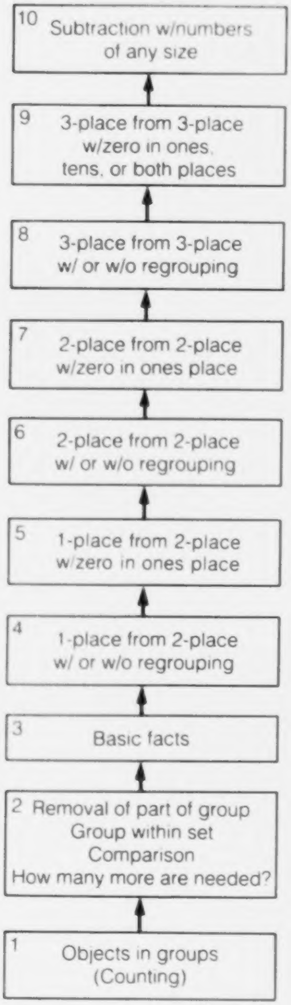
[6] Jean Piaget, *The Child's Concept of Number* (New York: The Humanities Press, 1952).

[7] Richard S. Skemp, "Mathematics as an Activity of Our Intelligence: A Model for Diagnosing and Remediation of Learning Difficulties in Mathematics," in Ian D. Beattie, ed., *1981 Research Monograph. Research Reports from the Seventh Annual National Conference on Diagnostic and Prescriptive Mathematics* (Bowling Green, Ohio: Research Council for Diagnostic and Prescriptive Mathematics, 1982), pp. 2–4; and Richard S. Skemp, *The Psychology of Learning Mathematics* (Hammondsworth, England: Penguin Books, 1971).

[8] See, for example, Robert M. Gagne, *The Conditions of Learning*, 3rd ed. (New York: Holt, Rinehart and Winston, 1977).

FIGURE 1-1

A sequence for learning to subtract whole numbers


10 Subtraction w/numbers of any size
9 3-place from 3-place w/zero in ones, tens, or both places
8 3-place from 3-place w/ or w/o regrouping
7 2-place from 2-place w/zero in ones place
6 2-place from 2-place w/ or w/o regrouping
5 1-place from 2-place w/zero in ones place
4 1-place from 2-place w/ or w/o regrouping
3 Basic facts
2 Removal of part of group
Group within set
Comparison
How many more are needed?
1 Objects in groups (Counting)

mathematics usually requires several years. Work with subtraction of whole numbers begins in kindergarten and is usually not fully accomplished until grade five or six.

Research Into Mathematics Teaching

Research into mathematics instruction has been conducted for many years. Although no single piece of research validates the claim of any "best way" to teach mathematics to children, the collective message garnered from many studies is that the use of manipulative materials during early stages of instruction is beneficial. Elizabeth H. Fennema has summarized research in a number of areas, including the use of Cuisenaire rods to help children understand place value. (Cuisenaire rods are pictured in Figure 6-1.) Research showed that use of these rods was superior to the absence of manipulative materials over the whole range of activities in the first grade. Fennema's review of research led her to conclude that the data support the following hypothesis: that a learning environment that is rich in manipulative models suited to the developmental level of the learner facilitates learning better than an environment that ignores the developmental level of the learner. She advocates the use of many materials in the primary grades, with a gradual decrease in their use as children are able to handle concepts more symbolically.[9]

Marilyn N. Suydam's summary of research shows that lessons that incorporate manipulative materials have a greater chance of producing greater learning than lessons not incorporating such materials. She also says that achievement is enhanced across a variety of topics and at every grade, achievement, and ability level.[10] The mathematical concepts presented in the middle grades are just as abstract and difficult for children there to understand as are the concepts presented to primary-age children. Work at advanced levels with whole numbers, as well as with common and decimal fractions and percent, measurement, geometry, and other topics, are all better understood when children use manipulative materials in appropriate ways.

Support from the Classroom

Classroom teachers who have used a meaningful approach to help children learn mathematics support the research evidence by attesting to the success of their materials and procedures. Mary Baratta-Lorton successfully used a wide variety of materials and procedures, which became the basis of books for primary-grade teachers.[11] In Minnesota a program based on the theories of Piaget and using a wide variety of manipulative materials has been successfully used in primary classrooms.[12] Marilyn H. Jacobson, a Pennsylvania mathematics consultant, describes three sets of materials to help middle-grade children learn about decimal fractions. The materials are used along with realistic problem situations to give the children a basis for understanding decimal fractions and their uses.[13]

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[9] Elizabeth H. Fennema, "Models and Mathematics," *Arithmetic Teacher*, XIX, No. 8 (December 1972), pp. 635–640.

[10] Marilyn N. Suydam, "Research Report: Manipulative Materials," *Arithmetic Teacher*, XXXI, No. 5 (January 1984), p. 27.

Learning theories and evidence from research in mathematics education and classroom practices support a mathematics program that is properly sequenced and rich in manipulative materials. Well-chosen, properly used materials enhance children's learning, generate interest, and promote problem-solving and computational skills.

USING RESEARCH AND KNOWLEDGE OF CLASSROOM PRACTICES

You will apply information learned from research and knowledge of classroom practices if you consider three equally important factors as you organize lessons and plan activities:

- Four levels of mathematical behavior
- A teaching-learning sequence
- A sequence of subtasks for a topic

Mathematical Behaviors

Four types of mathematical behavior are described here:

- Use of knowledge and facts
- Demonstration of understanding and comprehension

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[11] Mary Baratta-Lorton, *Mathematics Their Way* (Menlo Park, Calif.: Addison-Wesley, 1972); *Workjobs* (Menlo Park, Calif.: Addison-Wesley, 1972); *Workjobs II* (Menlo Park, Calif.: Addison-Wesley, 1978).

[12] A. Dean Hendrickson, *Mathematics the Piaget Way* (St. Paul: Minnesota Council on Quality Education, 1981).

[13] Marilyn H. Jacobson, "Teaching Rational Numbers—Intermediate Grades," *Arithmetic Teacher*, XXXI, No. 6 (February 1984), pp. 40–42.

- Application of mathematics in a variety of situations
- Analysis of relationships and patterns in nonroutine ways[14]

Facts and skills are the least cognitively complex of the four mathematical behaviors and are the easiest to select, teach, and evaluate. Examples of facts are $5 - 2 = 3$ and $9 - 9 = 0$, which are 2 of the 100 basic facts for subtraction. Skill is demonstrated by a child who uses the decomposition algorithm to compute answers for combinations like $63 - 14 = \square$ and $92 - 48 = \square$.

Understanding and comprehension comprise the second level of mathematical behavior. At this level of thinking, the child uses principles and concepts to explain the meaning of a fact or operation. Now a child can demonstrate with objects or explain with words one or more of the situations that give rise to a subtraction fact or a sentence like $46 - 27 = 19$. A child can also demonstrate with an appropriate place-value device and explain in words the meaning of the algorithm used to compute the sentence's answer.

The third level of mathematical behavior is application. Students achieve this goal when they are able to apply the facts and skills, concepts, and principles they have learned to everyday situations and to the solution of nonroutine problems. They know that subtraction is frequently applied in routine ways, such as when it is used to determine the difference in the capacities of two auditoriums. At other times the problem is such that the means of solution is not immediately obvious. Now students must analyze a situation to determine possible strategies and select the appropriate one(s). Students with a good understanding of subtraction will recognize when to use it to help solve nonroutine problems.

Analysis is the most cognitively complex of the mathematical behaviors. It requires that students engage in the type of abstract thinking associated with Piaget's fourth stage of mental development. It requires the detection of patterns and relationships and the organization of ideas into well-ordered, sometimes complex, structures. Analysis is demonstrated by a student who explains why the difference between an odd and an even number is always an odd number, and why the difference between two even or two odd numbers is always an even number. This level of mathematical behavior is not usually achieved by children until they have reached the upper grades of elementary school. There are many who do not achieve it at all while in elementary school.

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[14] These categories are adapted from Edward G. Begle and James W. Wilson, "Evaluation of Mathematics Programs," *Mathematics Education* (Chicago: National Society for the Study of Education, 1970), pp. 367–404. Other categories are found in Benjamin S. Bloom, ed., *Taxonomy of Educational Objectives, The Classification of Educational Goals, Handbook I; Cognitive Domain* (New York: Longmans, Green and Co., 1956); and Len Pikaart and Kenneth J. Travers, "Teaching Elementary School Mathematics: A Simplified Model," *Arithmetic Teacher*, XX, No. 5 (May 1973), pp. 332–342.

In 1977 the National Council of Supervisors of Mathematics viewed with alarm the emphasis that was being placed on work at the lowest level of mathematical behavior. They saw that instruction was being concentrated on facts and skills, while other types of behavior were being deemphasized.[15] Children whose work is confined largely to the lowest level of behavior can learn to compute accurately and rapidly, but they will possess few other mathematical behaviors and skills. All children should have experiences that lead them to the higher levels of behavior in all areas of mathematics.

A Teaching-Learning Sequence

Brownell's meaning theory stresses that children must understand what they are learning if learning is to be permanent, while Dienes, Piaget, and Skemp contend that children acquire understanding through the manipulation of objects and the internalization and structuring of concepts. All of these beliefs support the practice of beginning new work at a concrete-manipulative level of activity. This work, the first stage in the teaching-learning sequence, provides the mental images children need in order to perform the mental operations that lead to the internalization of concepts. The second stage in the sequence is the pictorial representation of objects. Now pictures of objects, rather than the objects themselves, are used to represent mathematical ideas. Later the work becomes more abstract when representations of objects, such as geometric shapes or dots, replace the pictures. Finally a totally abstract representation is presented through use of words and mathematical symbols.

The following description of a sequence of activities illustrates how first-grade children follow this sequence as they learn that subtraction can be used to compare the sizes of two sets of objects. The children have "math boxes," each of which contains similar objects. Each child is directed to count out a group of 8 blue objects and a group of 3 red objects. To determine how many more blue objects there are than red objects, each child matches each red object with one blue object until all of the red objects have been matched. The number of unmatched blue objects is the difference in the sizes of the two groups. Many activities with objects from the "math boxes" can provide the concrete experiences each child needs in order to understand how to use subtraction to compare sets of objects.

Later the children are given activity pages containing pictures of sets of objects similar to the ones in their "math boxes." Rather than making direct matchings, children draw lines to connect pictures in pairs of sets. The number of unmatched objects in the larger set is the difference in the size of the two sets. Similar pages containing representations of objects, such as red circles and blue squares, are used to provide further insight into this type of subtraction.

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[15] National Council of Supervisors of Mathematics, "Position Paper on Basic Skills," *Arithmetic Teacher*, XXV, No. 1 (October 1977), pp. 19–22.

Eventually subtraction sentences like $8 - 3 = 5$ are introduced to represent earlier actions with objects, pictures, and geometric shapes. The children's activities should also involve them in writing sentences and answers for situations such as these: "Joan has 17 baseball cards, while John has 8. Joan has how many more cards than John?" "Juan made 8 wreaths for the classroom; Tricia made 6. How many fewer wreaths did Tricia make than Juan?"

3 A Sequence of Steps

Gagne's research indicates that one way to improve learning is to determine a sequence of steps or subtasks that lead to the accomplishment of the target task and to teach the steps in order. Figure 1-1 shows one sequence of steps that might be used for ordering subtraction subtasks. While all children will not follow the sequence in exactly the same order, certain generalizations can be made about their progress through the subtasks. The subtasks that form the foundation for learning about and understanding of a terminal goal should be clear to every child. Before beginning the study of subtraction, for example, each child should be able to form sets of objects of specified sizes and to count the number of objects in preformed sets of real and pictured or representational objects. Thus, counting objects in sets is the first subtask for subtraction. Children need to know the kinds of situations that give rise to the operation. Step 2 of Figure 1-1 lists four kinds of subtraction situations.

All children need many experiences at the concrete level of operation with the "take-away" or removal-of-part-of-a-set situation, which is usually used to introduce children to subtraction, and the other three situations as they learn about subtraction. Once they have learned the meaning of an operation by working with objects, pictures, and representations of objects, they need to learn the basic facts for the operation. Then, once an operation is understood and its facts are learned, children are ready to move on to the higher level subtasks. The sequence of subtasks is less firm at higher levels because children will approach their study of them with different levels of mathematical knowledge and understanding. Some children will grasp new subtasks quickly and easily, with less attention to their order. Others will approach the new subtasks more slowly and deliberately and need a greater amount of time with each one.

With the background furnished by Chapter 1, you will now look at materials and procedures for assessing children's mathematics backgrounds and their interests in and attitudes toward the subject.

Summary

Emphasis is shifting from programs that stress memorization of facts and development of paper-and-pencil procedures to programs that stress problem-solving skills and the proper use of modern computing technology in elementary school classrooms throughout the country. It is widely recognized that children must develop skills that will make them lifelong learners so they can effectively cope with the unknown changes that will occur during their lifetimes. Every child needs a comprehensive mathematics program that focuses on problem solving, includes calculators and computers, and develops estimation and mental arithmetic skills. Such a program includes a study of numbers and operations, geometry, measurement, patterns and functions, probability and statistics, tables and graphs, and number theory.

Classroom activities that emphasize a teaching-learning progression begin with the use of objects that children can manipulate and count, then progress to pictures and representations of objects and finally, to words and symbols. Success of such a progression is supported by evidence from the research of Brownell, Dienes, Piaget, Skemp, and Gagne. Brownell introduced the meaning theory in 1935. Recently the theories of Dienes, Piaget, and Skemp have influenced the way mathematics is taught by showing teachers the importance of objects and models for developing understanding of mathematical concepts. Piaget and Skemp have also emphasized the importance of recognizing the stages of mental maturity through which each child passes. Gagne contributed the notion of tasks and subtasks for a topic. According to his theory, teachers must sequentially organize and teach the subtasks that lead to accomplishment of a final task so that children will have a firm foundation upon which to build their understanding of the final task. Research dealing with instruction in mathematics and the practices of successful teachers also support the use of manipulative materials with learners of all ages in the early stages of learning a topic.

Study Questions and Activities

1. Select a teacher's manual for a mathematics textbook for one elementary school grade. Examine its scope-and-sequence chart and compare the topics it contains with the seven contained in this chapter. List any topics contained in the elementary textbook that are not in this chapter; list any from this chapter not contained in the textbook. How do you account for any differences? Select one skill area from the scope-and-sequence chart and list its subtasks in the order listed in the chart. In what grade does the textbook begin instruction for the skill area you selected? At what grade is instruction concluded, if it is before the end of grade six?

2. Piaget's theories have been very influential in mathematics education for the last twenty years. Read from books and articles that interpret Piaget's theories of mental development to learn more about the four stages of cognitive growth and how they relate to learning mathematics. Articles and books included in the For Further Reading section at the end of this chapter will be useful.
3. Read Skemp's articles on instrumental and relational understanding (see For Further Reading section). Distinguish between the two types of understanding. Based on your experience with mathematics, which of these understandings have you developed? Give examples of either or both types of understanding you presently possess. Can you think of any situations in which instrumental understanding might be preferred over relational understanding?
4. Analyze the three classroom vignettes from this chapter. The three teachers represent a teaching-learning sequence that extends over several years. How do the lessons demonstrate this teaching-learning sequence and a sequence of skills? How do the teachers demonstrate understanding of different cognitive abilities? List specific examples of manipulative materials and activities used to involve students in the learning process.
5. Sue Brown and Donald Brown looked at teaching methods reflected in mathematics textbooks over this century (see For Further Reading section). Compare a recent textbook to a textbook twenty years or older. What differences do you find in content and approach?
6. "Back-to-basics" is a term often used to advocate a mathematics curriculum that emphasizes computation alone. The National Council of Teachers of Mathematics believes that basic skills in mathematics must include more than computation. Several articles listed in the For Further Reading section present discussions of back-to-basics and goals in mathematics. Read one of the articles and summarize the author's point of view. Talk to teachers, administrators, and parents about what they consider to be "basic" in a mathematics program. Do their comments indicate a narrow or broad view of what is basic?

For Further Reading

Adler, Irving. "Mental Growth and the Art of Teaching," *Arithmetic Teacher*, XIII, No. 7 (November 1966), pp. 576–584. A discussion of a dozen facets of Piaget's theory of mental growth and thirteen implications of the theory for teaching mathematics.

Arithmetic Teacher, XXXIII, No. 6 (February 1986). This theme issue focuses on manipulative materials in mathematics learning; it contains many articles on how to use concrete materials.

Arithmetic Teacher, XXVI, No. 6 (February 1979). The theme of this issue is a comprehensive mathematics curriculum. Probability, statistics, geometry, estimation, computers, and other topics important in mathematics are discussed.

Brodinsky, Ben. "Back to the Basics: The Movement and Its Meaning," *Phi Delta Kappan*, LVIII, No. 7 (March 1977), pp. 522–527. The "back-to-basics" movement that began in the late 1970s, including strengths, weaknesses, its supporters, educators' responses, and other points associated with the movement, are discussed.

Brown, Sue D., and Donald E. Brown. "A Look at the Past," *Arithmetic Teacher*, XXVII, No. 5 (January 1980), pp. 34–37. Teaching methods of the past century, as reflected in mathematics books, are briefly described. Excerpts from many books are presented.

Driscoll, Mark. "Effective Teaching," *Arithmetic Teacher*, XXXIII, No. 9 (May 1986), pp. 19, 48. Recent research indicates the factors that contribute to effective mathematics teaching.

Duckworth, Eleanor. "Piaget Rediscovered," *Arithmetic Teacher*, XI, No. 7 (November 1964), pp. 496–499. A discussion of Piaget's theories on intellectual development, teaching, and teacher training.

Dyrli, Odvard E. "Programmed Piaget," *Learning*, V, No. 2 (October 1976), pp. 78–80, 82, 84, 86. A 22-unit "program" helps the reader understand Piaget's four levels of cognitive development and their implications for the teacher. This article is recommended as good for the novice with regard to Piaget's theories.

Elkind, David. "What Does Piaget Say to the Teacher?" *Today's Education*, LXI, No. 8 (November 1972), pp. 46–48. Piaget's interview technique is used to identify some of the implications of his research for teachers. Three aspects of the technique and how they might affect teaching are reviewed.

Fennema, Elizabeth. "Manipulatives in the Classroom," *Arithmetic Teacher*, XX, No. 5 (May 1973), pp. 350–352. Seven reasons for using manipulatives, or learning aids, while teaching mathematics are discussed. The author also cautions against the use of manipulatives in a random way.

Gagne, Robert M. *The Conditions of Learning*, 3rd ed. (New York: Holt, Rinehart and Winston, 1977). A presentation of Gagne's latest views on how learning takes place. While this book deals with learning in general, many of the examples pertain to mathematics.

Gibb, E. Glenadine. "Teaching Effectively and Efficiently," *Arithmetic Teacher*, XXIX, No. 7 (March 1982), p. 2. Gibb discusses different interpretations of what efficiency and effectiveness in mathematics are. She concludes by saying that if changes in curriculum are to take place, the expected learning outcomes and expected content cannot be overlooked.

Greenwood, Jay, and Robert Anderson. "Some Thoughts on Teaching and Learning Mathematics," *Arithmetic Teacher*, XXXI, No. 3 (November 1983), pp. 42–49. Greenwood and Anderson discuss opinions about mathematics goals and methods, and they include suggestions for resolving conflicts.

Lovell, Kenneth R. "Intellectual Growth and Understanding Mathematics: Implications for Teaching," *Arithmetic Teacher*, XIX, No. 4 (April 1972), pp. 277–282. Lovell discusses the implications of Piaget's research for the mathematics educator.

Nasca, Donald. "Math Concepts in the Learner Centered Room," *Arithmetic Teacher*, XXVI, No. 4 (December 1978), pp. 48–52. Nasca identifies three stages in the acquisition of mathematical concepts. Premature emphasis on symbolism interferes with concept learning. Examples of topics presented in ways that stress understanding are illustrated.

National Council of Supervisors of Mathematics. "Position Paper on Basic Goals," *Arithmetic Teacher*, XXV, No. 1 (October 1977), pp. 19–21. There is a need for a clear definition of the goals of school mathematics. The goals must be broader than computational skills alone. The Council's position paper identifies ten basic skills to be developed by completion of high school and gives a rationale for a broad definition of basic skills.

National Council of Teachers of Mathematics. *An Agenda for Action, Recommendations for School Mathematics of the 1980s* (Reston, Va.: The Council, 1980). The Council makes eight recommendations that deal with content, instruction, evaluation of programs, amount of mathematics for students, teachers, and support for programs.

O'Hara, Ethel. "Piaget, the Six-Year-Old, and Modern Math," *Today's Education*, LXIV, No. 3 (September/October 1975), pp. 32–36. Piaget's four stages of cognitive growth are briefly reviewed. The focus is on preoperative first graders and what their level of cognitive development means for the mathematics program.

Piaget, Jean. *The Child's Conception of Number* (New York: The Humanities Press, 1952). Piaget describes, with many anecdotes regarding children's reactions, his research into how children think about numbers.

Skemp, Richard R. *The Psychology of Learning Mathematics* (Hammondsworth, England: Penguin Books, 1971). This book deals with how mathematics is learned rather than how it is taught. The first half looks at the thought patterns people adopt when they do mathematics, and analyzes them psychologically. The second half applies the psychology of the first half to the learning of mathematics, beginning with the child's first concepts of number.

———. "Relational Understanding and Instrumental Understanding," *Arithmetic Teacher*, XXVI, No. 3 (November 1978), pp. 9–15. Relational understanding indicates that the learner knows the relationships among parts—knows the schema—of a topic or concept, while instrumental understanding relies on rules only. The advantages of relational understanding outweigh those of instrumental understanding, according to Skemp, even though it is a harder type of understanding to develop.

Suydam, Marilyn N. "Manipulative Material and Achievement," *Arithmetic Teacher*, XXXIII, No. 6 (February 1986), pp. 10, 32. Suydam has summarized recent findings on the use of manipulative materials in elementary school mathematics.

Chapter

2

Assessment Procedures in Mathematics

In Chapter 2, you will read about

1. the importance of assessment in teaching mathematics
2. five kinds of readiness
3. techniques for evaluating children's skills and understanding of mathematics
4. conferences with parents
5. assessment of attitudes toward mathematics
6. a variety of tests designed to assess mathematical knowledge and understanding

The success of a mathematics program depends on how well it relates to children's levels of maturity, mathematical skills and understandings, and attitudes toward the subject. Current and accurate information about each child is essential before good planning and teaching can take place. There are five areas of readiness pertaining to mathematics and a variety of techniques for gathering information about each child's current status with regard to each area that you must understand and use as you develop and present lessons.

AREAS OF READINESS

In a paper delivered at the first National Conference on Remedial Mathematics in 1974, Robert B. Underhill named five areas or components of readiness that teachers need to consider as they plan and teach mathematics:

- Content Readiness
- Pedagogical Readiness
- Maturational Readiness
- Affective Readiness
- Contextual Readiness[1]

1 Content Readiness

Content readiness refers to the mathematical skills and knowledge that children possess. For example, a child who counts objects accurately, can demonstrate a "take-away" situation with cubes, knows all or most of the 100 basic subtraction facts, and understands place value in numbers from 9 to 99 has high content readiness for learning to compute answers for examples like these:

$$\begin{array}{r} 34 \\ -\ 6 \\ \hline \end{array} \qquad \begin{array}{r} 42 \\ -\ 9 \\ \hline \end{array} \qquad \begin{array}{r} 50 \\ -\ 7 \\ \hline \end{array}$$

Evaluation of children's daily papers plus interviews, observation of children at work, and diagnostic tests are sources of information about content readiness.

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[1] Robert B. Underhill, "Classroom Diagnosis," in John L. Higgins and James W. Heddens, eds., *Remedial Mathematics: Diagnostic and Prescriptive Approaches* (Columbus, Ohio: ERIC Center for Science, Mathematics, and Environmental Education, College of Education, Ohio State University, 1976), pp. 33–35. Used by permission.

Pedagogical Readiness

Work with a variety of materials—objects, pictures, representations of objects, and symbols—is advocated as the means by which children gain the experiences they need for understanding and using mathematics. Both specially designed commercial and teacher-made learning aids, such as Cuisenaire rods and beansticks, and incidental materials, such as small plastic airplanes and poker chips, are used. Pedagogical readiness concerns children's understanding of and skills with the learning aids they use as they study mathematics. For example, children are pedagogically ready to use a place-value device to learn the meaning of the algorithm in the margin if they can represent the numbers 45 and 207 and show how exchanges (regroupings) of tens for ones, ones for tens, and exchanges between other place-value positions are made with the device. This means that work with a device to develop understanding of it must precede use of the device to explain an operation.

$$\begin{array}{r} 207 \\ -\ 45 \\ \hline \end{array}$$

Children must also understand the meanings of pictures and representational drawings associated with mathematics. For example, to illustrate an addition situation, a primary-grade book may portray a set of animals joining a second set of animals. This picture does not serve its intended purpose if the child is unable to visualize the action. Likewise a drawing of a triangular prism is of little value to a fifth grader who is unable to visualize what a prism is and to construct a three-dimensional model from the drawing. You should be alert to the misinterpretations that children may make of pictures in textbooks and workbooks. Observing children working with learning aids and pictures, listening to their verbal contributions during lessons that use aids, pictures, and drawings, and interviewing are some of the ways you can determine children's pedagogical readiness.

Maturational Readiness

Maturational readiness refers to children's levels of mental maturity. Jean Piaget's research led him to conclude that each person passes through four stages of mental maturity:

- The sensorimotor stage (zero to two years)
- The preoperational stage (two to seven years)
- The concrete operations stage (seven to twelve years)
- The formal operations stage (twelve years and older)

The sensorimotor stage occupies the years of life prior to verbal communication. It is during this stage that the foundations for later mental growth and mathematical learning are developed.

The preoperational stage sees the onset of a child's first use of words and understanding that objects and symbols can be used to stand for other objects. A. Dean Hendrickson describes preoperational children:

> Children who are not yet concrete operational are most likely to make judgments about situations based on immediate appearances of things. . . . Preoperational thinkers tend to center on single features rather than to coordinate features, often making decisions based on one attribute and then switching to a different attribute. . . .
>
> (They) tend to focus on states and to ignore the transformations made in going from one state to another. In a similar fashion, the child's thinking is irreversible in that an opposite set of transformations needed to reverse a change, or a compensatory plan to undo something, cannot be conceptualized.
>
> In forming classes, or part-whole relations, the preoperational or transitional child finds it difficult to see that one class can be a subclass of a larger one. . . .[2]

It is not until near the end of this period that children reach a level of understanding about quantity and number that makes it possible to establish a lasting correspondence between the elements in two sets, even when the configurations of the two sets are changed. Now children know that the number of objects a set contains remains constant even if the objects' positions are shifted about. The onset of this level of maturity occurs during the early school years, kindergarten to grade two. Primary-grade teachers who are aware that all children may not yet have reached this level of maturity will not prematurely introduce operations that depend upon lasting correspondence, such as addition and subtraction.

Children are in the concrete operations stage during most of the elementary school years. Now children are able to form mental images of objects and to think in terms of the whole rather than parts. They can mentally integrate the parts to form a whole and separate a whole into its parts. Children who can perform these mental operations, as Piaget labeled them, have achieved what he called "reversibility." During this stage children become ready to deal with inverse operations, such as addition and subtraction, and to group and regroup numbers, such as while using the decomposition algorithm for subtraction.

The final stage of mental maturity, the formal-operations stage, begins in late elementary school or later. Children gradually become more abstract thinkers and are able to form hypotheses, analyze situations to consider all factors that bear on them, draw conclusions and test them against reality.

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[2] A. Dean Hendrickson, *Mathematics the Piaget Way* (St. Paul: Minnesota Council on Quality Education, State Department of Education, 1981), pp. 2–3. Used by permission.

You can observe children as they work, conduct interviews, note children's responses to questions, and give tests to determine each child's level of maturational readiness.

Affective Readiness

Mathematics anxiety, which is often exemplified by a strong fear and dislike for mathematics, has only recently been identified as an area of concern for teachers. Children's attitudes toward mathematics have a large bearing on how they approach the subject. Those whose attitudes are positive and who realize that knowledge of mathematics broadens occupational choices and enriches personal life have high affective readiness and are more likely to succeed than those whose affective readiness is low. (More information about mathematics anxiety is contained in Chapter 3.) Attitude assessment devices, observations of children at work, evaluation of work products, and interviews with children and parents or guardians are sources of information about affective readiness.

Contextual Readiness

Contextual readiness refers to how aware children are of the ways mathematics is used. Children who are aware of different real-world applications of mathematics possess a higher level of contextual readiness than those whose knowledge of applications is narrow. Children with a high level of contextual awareness are likely to realize the importance of mathematics, while those whose level is low are not as likely to place as much importance on the subject. You can improve children's contextual readiness by helping them see connections between the mathematics they learn and their in- and out-of-school activities. You can assess children's contextual readiness by observing their uses of mathematics both during times devoted to its study and when they are engaged in art, science, and other activities, and by interviewing children and parents.

ASSESSMENT PROCEDURES

You will need current information about each child's status with regard to the five areas of readiness as you prepare activities for your children. Written tests commonly come to mind when assessment procedures are considered. But while tests form one aspect of evaluation, they alone will not provide the comprehensive information you need to make a good match between children and their mathematics program. Along with tests, you should make systematic use of the following procedures:

Evaluation of Daily Work
Observation of Children
Interviews with Children
Evaluation of Children's Uses of Mathematics
Conferences with Parents or Guardians
Attitude Assessment

Evaluation of Daily Work

During the course of a year, children complete many work pages, make charts, tables, and graphs, and produce other work products. Some work is the result of an activity related to the introduction of a new topic; some work results from practice exercises dealing with algorithms and other skills; while some comes from activities associated with other subjects, such as science or art. A careful study of children's work products will give you information about their skills in performing routine tasks and how well they understand what they are doing.

An example of how examination of a daily paper can assist a child is provided by the following situation: A teacher introduced the decomposition algorithm for subtraction by having children use a place-value device to comprehend the regrouping process. Work with the device was followed by work with the algorithm modeled by the teacher and some of the children. He assigned a page from the textbook when he thought the children were ready for a practice exercise. He then examined the practice paper completed by each child. The following errors were commited on one paper:

$$\begin{array}{r} 42 \\ -19 \\ \hline 37 \end{array} \qquad \begin{array}{r} 67 \\ -48 \\ \hline 21 \end{array} \qquad \begin{array}{r} 85 \\ -36 \\ \hline 51 \end{array}$$

The teacher saw that the child subtracted the smaller number from the larger number regardless of where the numbers appeared in the algorithm. He knew immediately that the child did not understand the process, even though he had previously determined that the child was ready for the new work. Rather than allow the child to continue practicing the incorrect process until it became habitual, the teacher used a different device and additional modeling to review the regrouping process with the student. A practice page completed after the review revealed that the child now comprehended the decomposition process and was ready to complete other practice examples and use the process in problem-solving situations involving subtraction.

When a teacher finds that errors persist after reteaching a skill, it may be that the particular skill is too difficult at the time for the child and that an adjustment in the program is needed. Errors indicate misunderstandings, and detecting them early is the main reason for checking daily work products. (See pages 333–338 of Chapter 9 for examples of common errors in addition and subtraction of whole numbers.)

Other work products, such as the tables, charts, and graphs that children prepare, should be studied carefully to determine if there are errors in their construction or misinterpretations of the data they display.

Observation of Children

You will find that observation of each child during class discussions, tests, and work periods pays big dividends. Information is revealed about a child's interests in and attitudes toward mathematics, knowledge and understanding of the subject, and applications of mathematics in problem-solving situations. Observations need not be highly structured or time consuming. They can be made during class discussions, as children engage in seatwork activities, and as problems are solved. As children discuss a topic, note each child's contributions. Do some children tend to dominate the discussion? Are there some who never or infrequently contribute? What is the quality of each child's contribution? Does a child reveal the development of insights into the topic as it is discussed?

When children use learning aids, you should observe them carefully for clues that indicate how well they are developing an understanding of the aids and their uses. Are there children who use a device in a mechanical manner, applying it in an imitative way? Do others reveal that they understand the meaning of the device by applying it to new situations or in original ways? Do some children use several different devices to represent or illustrate a given process? Do some rely on learning aids more than their levels of maturity suggest they should?

You should observe children during tests and study periods to note work habits, use of materials, and ability to perform operations on numbers. When children begin work immediately and apply themselves consistently, you can usually conclude that they understand what they are doing. On the other hand, children who are slow getting to work or cannot keep their minds on it show that they may not understand the work. By circulating among the children as they work, you can observe those who are having difficulty and give them immediate assistance. Often a child can overcome difficulty in a short time with immediate help. If several children have a common problem, they should be formed into a small group for special assistance.

You should observe children from time to time as they use measuring instruments such as rulers, scales, and thermometers. If they use them incorrectly, they should get immediate help from you or another child.

Children should receive encouragement as they work. A soft-spoken "That's the way" or "You're on the right track" is often the only praise needed by children who are uncertain about whether they are working correctly.

Interviews with Children

An interview can be planned or spontaneous. Neither type of interview needs to be highly formal or time consuming. The following examples of planned interview situations illustrate their usefulness. When a new student transfers into a classroom, the child can be questioned about the work done in the former school and asked to demonstrate with a mathematical device or an algorithm what he or she understands about a given operation. While the interview will not reveal all of the information about the child's mathematical background that the teacher needs, it will provide an idea of where to start instruction for that child. When a teacher observes a student who has revealed a unique or special method of shortcutting a process or has an idea of how to approach a problem situation, she can use an interview to identify the child's thought processes in order to provide appropriate support for the process or idea. Another common use of the planned interview is with a student who is having progress problems so the teacher can identify appropriate materials and activities to help the child overcome the difficulties.

Interviews conducted to determine thought processes should not be threatening to a child. Simply ask the child to "think aloud" as the work is completed: "Tell me what you do when you complete this subtraction example." Do not interrupt or attempt to correct any faulty processes as the child works. Note errors and possible reasons for misconceptions that will help you understand why the incorrect process is being used.

Spontaneous interviews are used in situations like these: A child who understands a process is temporarily "blocked" and cannot continue the work. When the teacher discusses the situation with the child, she attempts to clarify the child's thinking and supports his or her own attempts to clarify the next steps. A teacher observes a child who is using time-consuming procedures, such as counting his fingers, and talks to the child to determine if there are more appropriate supports, such as a table of facts or a calculator, that will give the child a more efficient way to work. A teacher merely talks to a child for a few moments to give support, contact, or proximity approval. The child may not need either diagnosis or remediation but benefits from the teacher's closeness and comfort.

Evaluation of Children's Uses of Mathematics

Children's uses of mathematics in situations outside of mathematics lesson experiences reveal much about their understanding of operations, their ability to apply mathematics to the solution of everyday problems, and their attitudes toward the subject. First of all, teachers need to provide frequent opportunities for children to use mathematics to solve problems. For example, with your help children will learn how to use instruments to make accurate measurements and how to use charts and graphs to organize and report data for a social studies or science project. Later, you must be aware of your children's spontaneous uses of these and other mathematics skills in other situations so that you can give guidance and assistance to help them broaden their applications of mathematics both within and outside of the classroom. You should also help children evaluate their finished projects to determine if the charts, graphs, tables, or other work products serve their intended purposes.

Conferences with Parents or Guardians

Parents or guardians of a child can provide information that is often unavailable from other sources. The parents of a five- or six-year-old can reveal the extent to which the child has been exposed to activities that provide experiences with numbers. Many children enter kindergarten from a preschool where they were exposed to pattern blocks and counting materials, where they made simple graphs, learned to read and write some numerals, and engaged in other mathematics-oriented activities. Some young children have opportunities to use computers for prenumber and number activities before they enter kindergarten or first grade. Information about activities at preschools and with computers, as well as other less structured experiences that contribute to a child's mathematical background, will help you plan activities that are appropriate for your children.

Parents of older children can provide information about their children's exposure to mathematics via a computer, their uses of mathematics, attitudes toward the subject, and other significant information. A parent's personal anxiety about mathematics is frequently revealed during a conference. A parent may also talk about how important it is that the child excel in mathematics. Children respond to parents' attitudes and pressures to succeed in different ways. Some become anxious themselves or blame their failure on an "inherited" inability to learn mathematics. One child may respond to pressure with no problems, whereas another buckles under the pressure, fails, and dislikes mathematics. Children sometimes experience problems because of a conflict between the way a topic is taught in school and the way parents think it should be learned. Problems that arise because of parental attitudes, pressures, and ways of doing mathematics should be discussed openly and frankly during conferences. At the same time, parental help and support of work at home should be sought and planned.

Attitude Assessment

Success in mathematics is usually positively correlated with favorable attitudes toward the subject. Three devices used to determine children's feelings about mathematics—semantic-differential tests, attitude scales, and sentence completion forms—are discussed here.

a. Semantic-Differential Test A semantic-differential test contains a set of antonyms. Between each pair of antonyms is a seven-point scale on which an individual marks a point to indicate which word more closely represents that person's feeling about a subject. To prepare such a scale for mathematics, you would use a series of word pairs like these:

worthwhile . . . . worthless
easy . . . . difficult
boring . . . . challenging

Children mark the point that best expresses their individual feelings. A mark in the center can be scored 0. A plus 1, 2, or 3 is given to a mark in the direction of a positive expression, and a negative 1, 2, 3 is given to a mark in the direction of a negative expression. The greater the positive score, the more positive is a student's attitude toward mathematics; a score near zero indicates a neutral attitude; and a negative score shows a dislike for mathematics.

b. Attitude Scale An attitude scale is the most commonly used device for obtaining information about children's attitudes toward school subjects. The attitude scale in Figure 2-1 (p. 36) was designed to assess the attitudes toward mathematics of children in the intermediate grades and beyond. If you teach in a primary grade, you can reduce the number of items on this scale and simplify the wording so you can use the scale with your children. Give each child a response sheet that contains three "faces" for each item. Children indicate their feelings by circling a "happy," "neutral," or "sad" face as you read each item. A careful analysis of each child's responses to the scale will reveal whether or not the child has a strong like for, a neutral attitude toward, or a strong dislike for mathematics.

FIGURE 2-1
Mathematics attitude scale

Mathematics Attitude Scale

Directions: Please write your name in the upper right-hand corner. Each of the statements on this opinionnaire expresses a feeling or attitude toward mathematics. You are to indicate, on a five-point scale, the extent of agreement between the attitude expressed in each statement and your own personal feeling. The five points are: Strongly Disagree (SD), Disagree (D), Undecided (U), Agree (A), Strongly Agree (SA). Draw a circle around the letter or letters giving the best indication of how closely you agree or disagree with the attitude expressed in each statement.

1. I am always under a terrible strain in a mathematics class. SD D U A SA
2. I do not like mathematics, and it scares me to have to take it. SD D U A SA
3. Mathematics is very interesting to me, and I enjoy arithmetic and mathematics courses. SD D U A SA
4. Mathematics is fascinating and fun. SD D U A SA
5. Mathematics makes me feel secure, and at the same time it is stimulating. SD D U A SA
6. My mind goes blank and I am unable to think clearly when working mathematics. SD D U A SA
7. I feel a sense of insecurity when attempting mathematics. SD D U A SA
8. Mathematics makes me feel uncomfortable, restless, irritable, and impatient. SD D U A SA
9. The feeling that I have toward mathematics is a good feeling. SD D U A SA
10. Mathematics makes me feel as though I'm lost in a jungle of numbers and can't find my way out. SD D U A SA
11. Mathematics is something that I enjoy a great deal. SD D U A SA
12. When I hear the word mathematics, I have a feeling of dislike. SD D U A SA
13. I approach mathematics with a feeling of hesitation, resulting from a fear of not being able to do mathematics. SD D U A SA
14. I really like mathematics. SD D U A SA
15. Mathematics is a course in school that I have always enjoyed studying. SD D U A SA
16. It makes me nervous to even think about having to do a mathematics problem. SD D U A SA
17. I have never liked mathematics, and it is my most dreaded subject. SD D U A SA
18. I am happier in a mathematics class than in any other class. SD D U A SA
19. I feel at ease in mathematics, and I like it very much. SD D U A SA
20. I feel a definite positive reaction toward mathematics; it's enjoyable. SD D U A SA

Source: Lewis R. Aiken, Jr., "Research on Attitudes Toward Mathematics," *Arithmetic Teacher*, XIX, No. 3 (March 1972), copyright © 1972 by the National Council of Teachers of Mathematics. Used by permission.

c. Sentence Completion Form A sentence completion form is sometimes used to obtain information about a child's feelings. Rather than responding to prescribed items, the children are free to say whatever they want about a single topic or several topics. Items like these are used:

1. I think mathematics . . .
2. My mathematics class . . .
3. I like (or do not like) mathematics . . .

Items on a sentence completion form do not yield positive–neutral–negative scores. However, they may give a better reflection of a child's feelings than either the semantic-differential test or an attitude scale because of the freedom of expression they allow.

Scores on attitude scales are less valid in elementary school than in high grades. (Validity refers to the degree to which a score reflects a child's true feelings about a subject.) This is so because younger children have less stable attitudes and less insight into their feelings than older students.[3]

USING TESTS

Several types of tests are available. This section acquaints you with Piagetian-type, commercial achievement and diagnostic, and teacher-made diagnostic tests.

Piagetian-type Tests

During his research into how children learn, Jean Piaget devised a testing technique that he used to obtain information about children's mental maturity. Each test covered a specific topic, such as logical classification, ordering and seriation, time, or logical thought, and was given to one child at a time. Piaget's testing technique is helpful to kindergarten and first-grade teachers who want information about the mental maturity of their children.

An example of a Piagetian-type test is one designed to determine whether a child can make a one-to-one correspondence between objects in sets and can establish that if set A equals set B and set B equals set C, then set A equals set C (transitivity). This test can be administered to a child in 5 to 10 minutes. Give a child 3 collections of objects—perhaps 12 red plastic chips; 12 small, blue plastic blocks; and 12 small, green plastic cars. Line up 9 of the red chips and ask the child to make a set that contains as many blue blocks. If the child cannot establish the initial correspondence, allow further time for practice before you continue the test. Once a one-to-one matching has been established between the 2 sets, ask the child if she needs the remaining blocks. If the child uses the remaining blocks to fill in spaces, remove them before you continue. Ask: "Are there as many blocks as there are chips?" and "Why?"

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[3] Louis R. Aiken, Jr., "Research on Attitudes Toward Mathematics," *Arithmetic Teacher*, XIX, No. 3 (March 1972), p. 230.

Once the child's reply has been given, tell the child: "I am now going to spread out the blue blocks." Do this so that the line of blocks is longer than the line of chips. Ask: "Are there as many blocks as chips?" and "Why?" Children who have an immature understanding of one-to-one correspondence may argue that because the line of blocks is longer, there are more blocks; or that there are fewer chips because that line is shorter; or give some similar response. Children with a developing understanding of one-to-one correspondence will agree that the number of objects in the 2 sets is the same, but they will not be able to justify their answers. Children with a mature understanding will justify their answers by saying that it doesn't matter how long one line is, because there are the same number of blocks as before any changes were made; or that no objects were added or taken away; or some similar reply.

Children with a developing or mature understanding of one-to-one correspondence can go on to the second and third parts of the test. Move the set of red chips away from the blue blocks, and have the child establish a correspondence between the 9 blocks and 9 cars. Questions similar to those asked earlier determine whether the child can maintain the equivalence of the 2 new sets. To complete the test, ask the child if the number of red chips is the same as the number of green cars.[4]

In addition to the book *Checking Up I*, from which the above test was adapted, the Nuffield Foundation in England produced *Checking Up II* and *Checking Up III*, which also contain Piagetian-type tests. Richard W. Copeland's books, *How Children Learn Mathematics* and *Diagnostic and Learning Activities in Mathematics for Children*,[5] are useful sources of information about how to develop and administer Piagetian-type tests.

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[4] This test has been adapted from The Nuffield Foundation, *Checking Up I* (London: John Murray, 1970), p. 34.

[5] Richard W. Copeland, *How Children Learn Mathematics*, 3rd ed. (New York: Macmillan, 1979); and *Diagnostic and Learning Activities in Mathematics for Children* (New York: Macmillan, 1974).

Commercial Tests

Two types of commercially produced tests are commonly used in elementary schools—achievement and diagnostic. Achievement tests are designed to assess what children know about a subject or group of subjects, and are frequently used for state- or district-wide testing programs, where they are given to all children or those in selected grades according to a prescribed schedule. For example, the *SRA Achievement Test*[6] consists of a set of tests that provide information about reading, mathematics, and language arts in the primary grades; and about these three subjects plus social studies, science, and reference skills in the intermediate grades. Within an area, there are subtests dealing with different aspects of the subject, such as computation, problem solving, and concepts in mathematics. The single-subject *Tests of Achievement in Basic Skills—Mathematics Skills*[7] deals only with mathematics and contains subtests titled "Arithmetic Skills," "Geometry—Measurement—Application," and "Modern Concepts" with forms for grades K-1-2, 3-4, and 4-6 in the elementary school.

Commercially produced tests have standardized procedures for administering them, analyzing results, and determining a child's standing, which is usually stated in terms of a grade level or percentile ranking on each subtest and for the test as a whole. You can use a standardized achievement test as one source of information to determine a child's general level of understanding of the mathematics contained in the tests. The information will help you identify the children who have weaknesses that need in-depth diagnosis.

A textbook series usually contains tests that are designed to give a beginning-of-the-year inventory and end-of-unit and end-of-year mastery information about each child. These tests cover the mathematics content of the series for which they are designed. They can be used to determine where to place a child in a series and for screening children to determine which ones need in-depth diagnosis.

Achievement tests provide useful information, but they do not always yield sufficient details to identify a child's particular strengths and weaknesses. A comprehensive, analytical diagnosis is sometimes needed in order to obtain this information. Such an analytical diagnosis frequently includes a diagnostic test as one source of information. A diagnostic test is designed to provide information about strengths and weaknesses in specific areas of mathematics, such as the addition of whole numbers.

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[6] *SRA Achievement Series* (Chicago: Science Research Associates, 1978).

[7] *Tests of Achievement in Basic Skills—Mathematics Skills* (San Diego: Educational and Industrial Testing Service).

A commercial diagnostic test will be useful as long as its items match those in your mathematics program. The *Keymath Diagnostic Arithmetic Test*[8] consists of a large book containing one item per page to test an increasingly difficult range of mathematics concepts and a page each of addition, subtraction, multiplication, and division algorithms for the child to solve. The test is administered to one child at a time. The test administrator shows the student a page from the test book, gives oral instructions, and receives the child's oral response. A written record of responses is made on a sheet provided with the test. In addition to oral responses, the child writes answers on the separate paper to demonstrate skill in computation.

The *Test of Early Mathematics Ability*[9] identifies specific strengths and weaknesses in mathematical thinking. The test is designed for children from four to almost nine years of age and covers both formal mathematics concepts—number facts, calculation, and base-ten concepts—and informal concepts—counting, informal calculations, and more and less. There are fifty questions with multiple items under each one that take about twenty minutes to administer to one child. Questions are asked by the administrator, and responses are given orally or written calculations are recorded in a test booklet.

The *Stanford Diagnostic Achievement Test*[10] is an example of a commercially developed group-administered diagnostic test. A child receives a test booklet that contains the test items and records a written response for each item. There is a test for each of four grade ranges: 1.6–4.5, 3.6–6.5, 5.6–8.5, and 7.6–13. Subtests deal with number systems and numeration, computation, and application.

The chapter pretests that come with most textbook series are criterion-referenced tests, given before any work on a chapter is begun, to diagnose children's understandings of and skills with the chapter's content.

Teacher-made Tests

There may be times when you do not have a suitable commercial or textbook diagnostic test to use. When this is true, you will need to prepare your own test. These four steps will help you prepare and use a test:

Choose the test's topic(s).
Choose the objectives.
Select the procedures and design the test.
Analyze the results.

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[8] *Keymath Diagnostic Arithmetic Test* (Circle Pines, Minn.: American Guidance Services, 1976).

[9] *Test of Early Mathematics Ability* (Austin, Tex.: PRO-ED, 1983).

[10] *Stanford Diagnostic Achievement Test* (New York: Harcourt Brace Jovanovich, 1976).

a. Kindergarten/First-grade test If you are a kindergarten or first-grade teacher, you will select topics such as one-to-one correspondence; rote and rational counting to 20 (or 50, or 100); numeral recognition; matching numerals with sets; classifying objects by size, shape, and color; and comparing objects by length, volume, and size. Most of these topics must be tested on a one-to-one basis, so you will select the number of topics you can test in about 10 minutes. For one test session, you might choose the first four topics from the list above.

Once you have chosen your topic(s), select performance objectives that are appropriate for the children. These may come from an already-prepared set of objectives, such as ones furnished by your school district or from a textbook teacher's manual, or from a list that you prepare yourself. You also need to choose the types of mathematical behaviors the children are to demonstrate. Objectives 2, 3, and 4 in the following list deal with facts and skills, and objectives 1 and 5 deal with understanding and comprehension from the list of behaviors discussed on pages 18–20.

1. Given a collection of up to 10 objects arranged in a straight line, the learner will make a one-to-one matching using similar objects.
2. The learner will count orally to 20.
3. Given a collection of 20 objects, the learner will count them.
4. When shown cards containing the numerals 0 through 9, the learner will say the number named on each one.
5. Given pictures of 1 to 9 objects and numeral cards for the numbers 1 to 9, the learner will match each picture with the numeral card that tells how many objects are in it.

After a topic and the level of mathematical behavior have been selected, you are ready to prepare materials for the test. To test performances on the five kindergarten–first-grade objectives, these procedures can be used:

1. Arrange 6 red poker chips in a straight line with a 1-inch space between each chip. Give the child 6 blue chips with instructions to match the blue chips with the red chips. Demonstrate what a match is, if necessary. Repeat with both larger and smaller sets of chips.
2. Tell the child to count to 20.
3. Arrange 20 red chips on one side of a desk or large sheet of paper (Figure 2-2, p. 42), and tell the child to count while moving them one at a time to the other side.

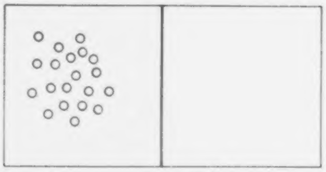


FIGURE 2-2

Set-up for testing a child's ability to count

4. Show one at a time and in mixed order a set of cards, each containing one of the numerals 0–9. Ask the child to tell you the number named on each card.
5. Use dot cards, like those in Figure 2-3, and numeral cards. Arrange the dot cards in random order in a line. Tell the child to match each numeral card with the correct dot card.

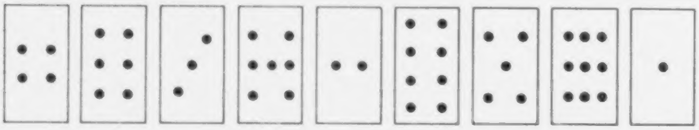


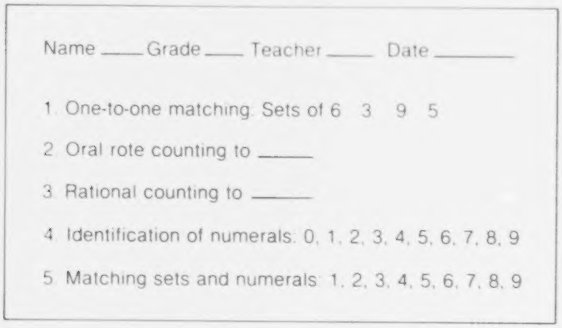
FIGURE 2-3

Cards for testing a child's ability to match quantity and numeral

You can record each child's response on a prepared record card. A 4-by-6 card like the one in Figure 2-4 is useful for this test. Circle numerals that indicate responses for numbers 1, 4, and 5. Write in numerals that tell how high each child counts in procedures 2 and 3.

Tests will provide information that will help you determine which children need more work with the topics included in the test's objectives and which are ready to move on to other objectives. The children's performances will also help you select materials and procedures for presenting topics.

b. Intermediate-grade test Level 4 of the subtraction sequence in



Name ____ Grade ____ Teacher ____ Date ______

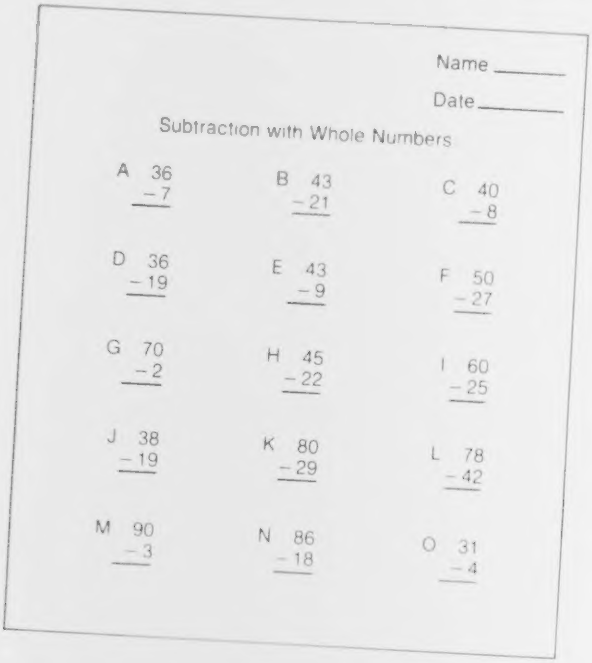
1. One-to-one matching: Sets of 6 3 9 5
2. Oral rote counting to ____
3. Rational counting to ____
4. Identification of numerals: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9
5. Matching sets and numerals: 1, 2, 3, 4, 5, 6, 7, 8, 9

FIGURE 2-4

Card for recording responses for individually administered test

FIGURE 2-5

Sample subtraction diagnostic test



Name ________

Date ________

Subtraction with Whole Numbers

A $\begin{array}{r}36\\-7\\\hline\end{array}$	B $\begin{array}{r}43\\-21\\\hline\end{array}$	C $\begin{array}{r}40\\-8\\\hline\end{array}$
D $\begin{array}{r}36\\-19\\\hline\end{array}$	E $\begin{array}{r}43\\-9\\\hline\end{array}$	F $\begin{array}{r}50\\-27\\\hline\end{array}$
G $\begin{array}{r}70\\-2\\\hline\end{array}$	H $\begin{array}{r}45\\-22\\\hline\end{array}$	I $\begin{array}{r}60\\-25\\\hline\end{array}$
J $\begin{array}{r}38\\-19\\\hline\end{array}$	K $\begin{array}{r}80\\-29\\\hline\end{array}$	L $\begin{array}{r}78\\-42\\\hline\end{array}$
M $\begin{array}{r}90\\-3\\\hline\end{array}$	N $\begin{array}{r}86\\-18\\\hline\end{array}$	O $\begin{array}{r}31\\-4\\\hline\end{array}$

Figure 1-1 on page 16 provides the subject for performance objectives at the facts and skills, understanding and comprehension, and application levels of mathematical behavior for an intermediate-grade assessment.

1. Given a subtraction combination like $43 - 9 = \square$, the learner will determine the answer using paper and pencil.
2. Given a subtraction combination like $43 - 9 = \square$, the learner will give an oral explanation of the steps involved in using the decomposition algorithm to determine the answer.
3. Given a subtraction combination like $43 - 9 = \square$, the learner will write an appropriate word problem.

Paper-and-pencil tests are commonly used to test children's knowledge of facts and skills. The test in Figure 2-5 contains 15 items, 3 for each of 5 different types of combinations from the subtraction sequence on page 16. No more than 15 items are needed. It is assumed that a child who gets 2 or all 3 of 1 type correct has developed computational skill with that type of subtraction. On the other hand, a child who misses 2 or all 3 may need further work with that type of subtraction. Note that the items are mixed on the page and not grouped by type. This forces a child to think about each separate item rather than rely on a pattern to complete the second and third of a type after the first one is completed. A wide variety of basic facts is included in the 15 items. Do not set a time limit for completing a diagnostic test; give each child all the time needed to complete it.

$$\begin{array}{r} 81 \\ -\ 26 \\ \hline \end{array}$$

An interview is better than a test to determine a child's understanding and comprehension of a subtraction algorithm. Give an example like the one in the margin, and ask the child to explain the work as it is completed. Review the suggestions for conducting an interview on page 33.

One way to test a child's routine application of subtraction in a practical situation is to present a number sentence like 256 − 149 = 107 and ask the child to make up a story problem about the sentence. A test containing a mixture of printed story problems requiring addition and subtraction, or a combination of other operations, can also be used to test a child's knowledge of how operations are used in routine problem situations.

The purpose of a diagnostic test is to determine a child's strengths and weaknesses in mathematics. Therefore it is necessary that you carefully analyze responses for each test you administer. The analysis requires attention to more than the number of items missed. You must note such things as the type of examples that are done correctly or incorrectly and any patterns of errors that may occur. In addition, you should determine whether any basic fact combinations are regularly answered incorrectly.

Teachers who regularly and systematically use the assessment procedures discussed in this chapter and who use the results of their evaluations to select suitable materials and develop appropriate activities will provide children with the mathematical skills and knowledge they need to meet the future with confidence.

Summary

Teachers need current information about each child in order to plan effective lessons and activities. Each child's content, pedagogical, maturational, affective, and contextual readiness should be assessed before planning is done. Content readiness refers to knowledge and understanding of mathematics concepts and skills. Pedagogical readiness refers to knowledge and understanding of materials—manipulative, pictorial, and representational drawings—that are used to teach mathematics to children. Maturational readiness is concerned with the mental maturity of children. The attitudes of children toward mathematics are the concerns of affective readiness. And the knowledge of the uses of mathematics that children possess is the subject of contextual readiness.

A variety of assessment procedures are used to determine readiness in the five areas. Evaluation of daily work, observation of children working with mathematics in the classroom, interviews with children, evaluation of children's uses of mathematics, and parent conferences reveal information about one or more areas of readiness that cannot be obtained from tests. Semantic differential tests, attitude scales, and sentence completion forms are used to obtain information about children's attitudes toward mathematics. Piagetian-type tests are based on the interview technique used by Jean Piaget; they provide information about a child's level of maturational development. Commercial achievement tests provide information about a student's general achievement in mathematics, while commercial diagnostic tests will provide more specific information about a child's strengths and weaknesses. Teacher-made diagnostic tests also reveal information about strengths and weaknesses and should be used when commercial tests with content that matches the teacher's objectives are unavailable.

Study Questions and Activities

1. Conduct a brief study of some aspect of kindergarten or first-grade children's concepts of mathematics such as suggested in this chapter. You may also use Copeland's books (see footnote 5 on page 38) as a guide for preparing your interview. Which level of understanding has each child attained? How can teachers at the primary level use Piaget's ideas and investigative techniques to improve their teaching?
2. Examine a commercial diagnostic test and its manual. What specific mathematical skills and understandings does it diagnose? Critique the test in terms of the attention it gives to facts and skills, comprehension and understandings, and applications of mathematics. Do you believe there is a good balance?
3. Look at the testing material available with a contemporary textbook series. Which specific mathematical skills and understandings are assessed? Compare the test to a commercial diagnostic or achievement test. How well do the two agree on content, vocabulary, and notation? What difficulties might arise for children when a test and a book do not use the same vocabulary and notation?
4. Use this chapter's suggestions for preparing a teacher-made diagnostic test. Select a topic and a sequence of 5 subtasks, then write appropriate objectives and items and design your test. Administer the test to a small group of children and analyze the results. What skills or subskills seem to be understood? How does this knowledge help to modify your teaching?
5. Ashlock's book and several articles found in the For Further Reading section contain examples of many error patterns found in student work. Examine several of the error patterns. Give problems similar to the examples to several students. Do you find any error patterns in their work? As you read further in this book, list techniques which might be useful in overcoming the misunderstandings shown in error patterns.

For Further Reading

Arithmetic Teacher, XXVII, No. 3 (November 1979). This theme issue contains eight articles on assessment.

Ashlock, Robert B. *Error Patterns in Computation*, 3rd ed. (Columbus, Ohio: Charles E. Merrill, 1982). This semiprogrammed book gives the reader the opportunity to study error patterns and establish procedures for helping children overcome them.

Brumfield, Robert D., and Bobby D. Moore. "Problems with the Basic Facts May Not Be the Problem," *Arithmetic Teacher*, XXXIII, No. 3 (November 1985), pp. 17–18. Brumfield and Moore describe the kind of error patterns students may make that prevent them from solving problems.

Carpenter, Thomas P., et al. "Student's Affective Responses to Mathematics: Results and Implications from National Assessment," *Arithmetic Teacher*, XXVIII, No. 2 (October 1980), pp. 34–37, 52–53. The results of the second National Assessment of Educational Progress section on attitudes indicated that a majority of 9-year-old children liked mathematics. The article discusses children's responses and presents major findings from the survey.

Clark, H. Clifford. "How to Check Elementary Mathematics Papers," *Arithmetic Teacher*, XXXIV, No. 1 (September 1986), pp. 37–38. Clark recommends that teachers use evaluation of daily work for diagnostic purposes.

Corbitt, Mary Kay. "When Students Talk . . . ," *Arithmetic Teacher*, XXXI, No. 8 (April 1984), pp. 16–20. Interviewing students gives the teacher an idea of mathematics from the students' point of view.

Engelhardt, Jon M. "Using Computational Errors in Diagnostic Teaching," *Arithmetic Teacher*, XXIX, No. 8 (April 1982), pp. 16–19. Engelhardt discusses four types of errors: mechanical, careless, conceptual, and procedural. Teachers must attend to each type of error and not concentrate on procedural errors, which they commonly do according to the author.

Fowler, Mary Anne. "Diagnostic Teaching for Elementary School Mathematics," *Arithmetic Teacher*, XXVII, No. 7 (March 1980), pp. 34–37. Diagnostic teaching requires three things: determination of types of errors, rationale for each type of error, and establishment of what needs to be learned in order to overcome each type of error. The article discusses diagnostic teaching in the elementary school.

Freeman, Donald J., et al. "A Closer Look at Standardized Tests," *Arithmetic Teacher*, XXIX, No. 7 (March 1982), pp. 50–54. Four standardized tests are discussed. The authors found considerable variation in their contents. A test should be selected carefully to provide the closest possible match between its content and a program's content.

Hodges, Helene L. B. "Learning Styles: Rx for Mathophobia," *Arithmetic Teacher* XXX, No. 7 (March 1983), pp. 17–20. Identification of different learning styles in children may help a teacher aid children to overcome anxiety.

Michaels, Linda A., and Robert A. Forsyth. "Measuring Attitudes Toward Mathematics? Some Questions to Consider," *Arithmetic Teacher*, XXVI, No. 4 (December 1978), pp. 22–25. The authors identify a number of factors to consider when preparing to test children's attitudes toward mathematics. They discuss points related to how to collect data, the attitudes to assess, types of items and their uses, length of scale, and validity.

Miller, Geoffrey. "Graphs Can Motivate Children to Master Basic Facts," *Arithmetic Teacher*, XXXI, No. 2 (October 1983), pp. 38–39. Miller describes how a teacher uses graphs to record improvement on basic facts.

Moyer, Margaret B., and John C. Moyer. "Ensuring That Practice Makes Perfect: Implications for Children with Learning Disabilities," *Arithmetic Teacher*, XXXIII, No. 1 (September 1985), pp. 40–42. Keep students from practicing mistakes by helping them notice types of problems.

Norton, Mary Ann. "Improve your Evaluation Techniques," *Arithmetic Teacher*, XXX, No. 9 (May 1983), pp. 6–7. Norton suggests alternative ways a teacher can use to evaluate a child's understanding of mathematics.

Rudnitsky, Allan N., Priscilla Drickamer, and Roberta Handy. "Talking Mathematics with Children," *Arithmetic Teacher*, XXVIII, No. 8 (April 1981), pp. 14–16. Presents guidelines and strategies for conducting interviews about mathematics.

Sadowski, Barbara R., and Delayne Houston McIlveen. "Diagnosis and Remediation of Sentence-Solving Error Patterns," *Arithmetic Teacher*, XXXI, No. 5 (January 1984), pp. 42–45. Analysis of student papers reveals misunderstandings and suggests activities for remediation.

Scheer, Janet K. "The Etiquette of Diagnosis," *Arithmetic Teacher*, XXVII, No. 9 (May 1980), pp. 18–19. Presents 12 guidelines to proper etiquette in diagnosis.

Shaw, Robert A., and Philip A. Pelosi. "In Search of Computational Errors," *Arithmetic Teacher*, XXX, No. 7 (March 1983), pp. 50–51. Observations and student interviews reveal the nature of students' computational errors. Four situations are discussed.

Sowder, Larry, Judith Threadgill-Sowder, Margaret B. Moyer, and John C. Moyer. "Diagnosing a Student's Understanding of Operations," *Arithmetic Teacher*, XXXIII, No. 9 (May 1986), pp. 22–25. These authors provide evaluation techniques for determining meanings of operations.

Suydam, Marilyn N. "Attitudes toward Mathematics," *Arithmetic Teacher*, XXXII, No. 3 (November 1984), p. 12. Suggests how teachers can improve attitudes toward mathematics.

Chapter

3

Planning for Teaching Mathematics

In Chapter 3, you will read about

1. a seven-step model for lesson planning
2. ways a learning center can be organized and used to extend instruction
3. advantages and problems with individualized instruction
4. mathematics laboratories and the role they can serve in mathematics instruction
5. the issue of equity in mathematics
6. mathematics anxiety and strategies for avoiding and lessening mathematics anxiety
7. characteristics of children with learning handicaps and suggestions for adapting mathematics instruction for them
8. characteristics of gifted and talented children and ways to enrich the mathematics curriculum for them

"The classroom teacher continues to be the gatekeeper in curriculum—the one who stages the learning opportunities."[1] The ways you stage opportunities should be based on theories of how children learn, research into mathematics education, successful classroom practices, the nature and structure of mathematics, and your knowledge of the children you teach. In order to stage the best learning opportunities, you must understand and use a repertoire of teaching procedures that maintains balance among different ways of guiding children's learning. The procedures discussed in this chapter are teacher-directed lessons, learning centers, individualized instruction, and mathematics laboratories. Equity in mathematics, mathematics anxiety, and special students also have their impact on how you teach mathematics. These topics are also included in this chapter.

TEACHER-DIRECTED LESSONS

In a teacher-directed lesson, a teacher works directly with a group of children to conduct a day's lesson. The lesson may be based on the content of a mathematics textbook series, a set of district or school objectives, or some other scheme for organizing each grade's mathematics program. When you use a textbook series as the basis for your program, you must avoid the habit of teaching each lesson by having children open their books to a lesson's first page and follow along as you explain concepts and processes. This practice leads to a static program that becomes uninspiring. A lack of variety can breed negative attitudes and cause some children to look at mathematics with distaste and as unworthwhile. Rather, you should view a textbook as *one* of the tools you use while teaching mathematics. A textbook cannot provide manipulative experiences for children during a lesson. It should not be used until you are certain that your children have had the manipulative and other experiences that will make the pictures, diagrams, and discussions contained in a book clear to them. The teacher's manual for each grade frequently suggests procedures and materials for activities to precede or follow use of the text.

A lesson based on a text that combines manipulation of learning aids, modeling by the teacher or selected students, discussion, and oral and written practice from the book is a practical way to guide children's learning of a new topic. Successful teachers know that well-developed plans are the foundation for effective lessons and activities. The lesson-plan scheme presented here is based on a format developed by Madeline Hunter and Douglas Russell, both of the Laboratory School at the University of California at Los Angeles. The plan separates a lesson into seven parts:

---

[1] Ad Hoc Mathematics Framework Committee, *Mathematics Framework and the 1980 Addendum for California Public Schools Kindergarten Through Grade Twelve* (Sacramento: California State Department of Education, 1982), p. 100.

- Setting the stage
- Statement of objective
- Instructional input
- Modeling
- Checking for understanding
- Guided practice
- Independent practice

Before you plan a lesson, you should do these things:

- Select a topic, such as subtraction with whole numbers.
- Be aware of the subtasks that lead to a major goal and of your children's understanding of them. In the lesson discussed here, the subtasks for subtraction with whole numbers from Figure 1-1 are used.
- Determine the level of mathematical behavior you want to stress—facts and skills, understanding and comprehension, application, or analysis.
- Determine each child's status with regard to readiness for the lesson—cognitive, affective, pedagogical, maturational, and contextual.

Assume that you are ready to introduce subtraction of a two-place number from a two-place number (subtask 6 in the sequence in Figure 1-1). Stated in performance terms, your objective might be, "The student will use beansticks to demonstrate the meaning of the decomposition algorithm for subtracting an example like $64 - 45 = \square$."

Setting the Stage

You want to develop the children's mental readiness for the lesson that is to come. Hunter and Russell use the term *anticipatory set* to refer to this activity's purpose. You shift children's attention from school activities that preceded mathematics, such as reading or recess, so they can focus on the lesson. You may also use this stage of the lesson to make children aware of the day's topic. One way to set the stage is to present a story problem in a familiar setting. "Yesterday I counted 42 peanuts in my peanut jar. So far today I have given out 18 peanuts. Can you tell me how many peanuts are in the jar?" Hold up a clear glass jar that contains 24 peanuts. Allow time for the children to think about the situation and determine the answer, if possible. Write the answer or answers, if more than one is given, on the chalkboard. Another way to set the stage would be to have children use their beanstick sets to show numbers you dictate, such as 62, 41, and 87. (Beansticks are pictured in Figure 7-4 and described on page 198.) Another way to set the stage is to play the "What's My Rule" game, which is explained on page 307, to review basic fact combinations with minuends that are between 9 and 19. The first activity uses a problem situation that is directly related to the topic of the lesson, while the other two require the children to demonstrate skill in using a learning aid or knowledge of basic facts that will be used during the lesson.

2 Statement of Objective

This step in the lesson acquaints children with the specific objective. "Today you are going to use beansticks to learn how to do this kind of subtraction":

$$\begin{array}{r} 42 \\ \underline{-18} \end{array}$$

3 Instructional Input

You provide direct instruction about the new work. You might use a sequence of questions and answers to show children how beansticks are used:

- "Use your beansticks to show the number of peanuts that were in the jar this morning." (Each child represents 42 with beansticks.)
- "Who remembers how many peanuts I have given out this morning?" Recall that 18 peanuts were given out, so you want to take away 18.
- "How can I take away 18 beans when there are not enough loose beans to take away 8 of them?"
- Guide the children to see that 1 tens stick can be exchanged for 10 loose beans. When this is done, there will be 3 tens sticks and 12 loose beans.
- "Can I take away 8 beans now?"
- Help the children recognize that they can take away 8 loose beans.
- "We have taken away 8 beans. How many more do we need to take away?"
- Help the children see that 1 tens stick can be taken away.

- Count the number of beans that are left.
- Repeat with other story problems until the children demonstrate that they understand the regrouping process.

4 Modeling

Modeling is a demonstration of how an operation or other mathematical process is completed. Use a set of enlarged beansticks to repeat the story problem example. Discuss each step in the process while the children watch. Repeat with other examples, such as 56 − 38 = ☐ and 43 − 12 = ☐. Help the children distinguish between situations where regrouping is necessary and where it is not as you model these two combinations.

5 Checking for Understanding

$$\begin{array}{r}45\\-26\\\hline\end{array}$$

Now you want each child to demonstrate the regrouping, when necessary, and the removal of beans to portray several subtraction combinations. Show an example like the one in the margin for each child to solve with beansticks. Check each child's work for accuracy before another example is begun. If there are children who obviously cannot use the beansticks accurately, reteach the lesson. Whether you do it immediately or on another day will depend upon your assessment of the extent of the misunderstanding and possible reasons for it.

6 Guided Practice

Guided practice follows checking for understanding. You can provide guided practice for this activity by having pairs of children work together; you present new combinations of numbers and the children demonstrate how to use the beansticks to each other. You will want to assess each child's understanding of the lesson's topic, so you should move among the children as they work to commend those whose work is accurate and discuss any faulty work.

7 Independent Practice

Independent practice is assigned following guided practice. Independent practice is not a part of this lesson because there is no practical way to assess a child's accuracy while manipulating beansticks other than by direct observation. Independent practice will come on subsequent days when the child learns to use the decomposition algorithm for subtraction with whole numbers. Then practice pages in the textbook or other paper-and-pencil materials will be used.

It is important that work completed during independent practice be carefully analyzed so you can determine, if possible, reasons for any errors that may occur. If the practice has involved computational processes, you should study the work paper for errors involving basic facts and for any patterned errors that might have occurred. Other work products, such as tables and graphs, need to be carefully analyzed as well. Once you are convinced that a child understands a new process, provide further practice to help make the learning permanent. The following considerations will promote successful practice sessions:

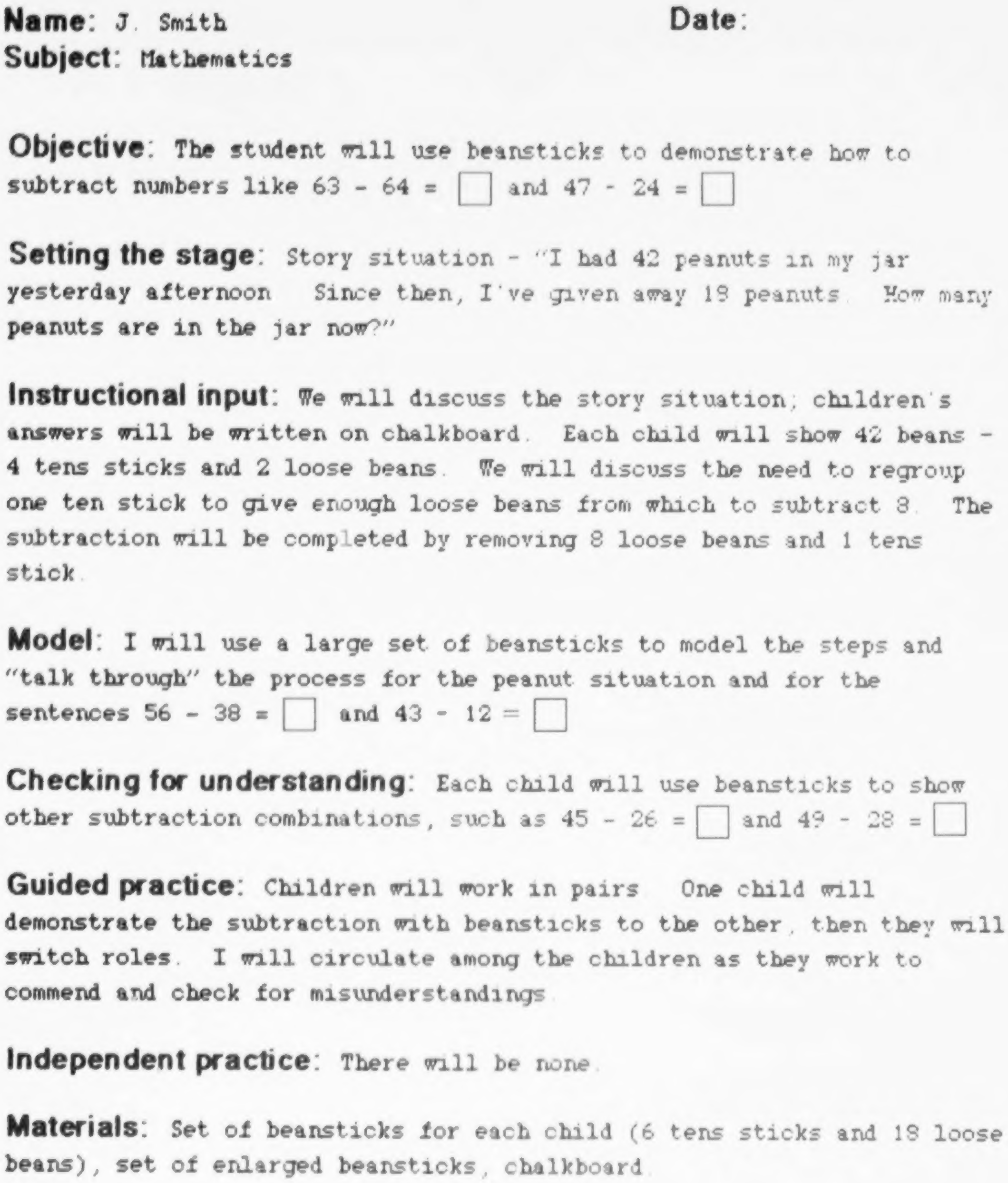
> The kind and amount of practice are not likely to be the same for all children at all times.
>
> A variety of materials and procedures should be used.
>
> Practice sessions should be brief and occur regularly.
>
> Once a skill is learned, review practice sessions should be given periodically.

Each of the seven steps is not *always* a part of every lesson. One or more steps may be omitted because they are inappropriate. Independent practice, for example, was not included in the lesson just described. Another example of where one or more steps would be omitted is a lesson that introduces a totally new topic. A lesson dealing with percent in a fifth- or sixth-grade class may not get beyond the modeling stage the first day. Activities with flats and cubes from a Cuisenaire set provide visual images to help children understand percent and occupy the entire period. On the second day, instructional input or modeling may not be needed because the lesson is a continuation of the previous day's work. On this day you set the stage and review the objective, then move on to a review of the previous day's work and the check-for-understanding part of the lesson. A sample lesson plan is shown in Figure 3-1a (on p. 54). It outlines the seven steps to guide your teaching and contains notes for the subtraction lesson. A blank lesson plan form is provided in Figure 3-1b (on p. 55) so that you may record your own notes for teaching a lesson.

LEARNING CENTERS

A learning center is a place within the classroom where materials and devices used for learning about mathematics are located. It may be permanently located on a table in a corner of the room or temporarily situated on a child's desk or worktable. A permanent center may contain learning aids such as primary-grade "math boxes," sets of Cuisenaire rods and boxes of

(text continues on p. 56)



Name: J. Smith **Date:**

Subject: Mathematics

Objective: The student will use beansticks to demonstrate how to subtract numbers like 63 - 64 = ☐ and 47 - 24 = ☐

Setting the stage: Story situation - "I had 42 peanuts in my jar yesterday afternoon. Since then, I've given away 18 peanuts. How many peanuts are in the jar now?"

Instructional input: We will discuss the story situation; children's answers will be written on chalkboard. Each child will show 42 beans - 4 tens sticks and 2 loose beans. We will discuss the need to regroup one ten stick to give enough loose beans from which to subtract 8. The subtraction will be completed by removing 8 loose beans and 1 tens stick.

Model: I will use a large set of beansticks to model the steps and "talk through" the process for the peanut situation and for the sentences 56 - 38 = ☐ and 43 - 12 = ☐

Checking for understanding: Each child will use beansticks to show other subtraction combinations, such as 45 - 26 = ☐ and 49 - 28 = ☐

Guided practice: Children will work in pairs. One child will demonstrate the subtraction with beansticks to the other, then they will switch roles. I will circulate among the children as they work to commend and check for misunderstandings.

Independent practice: There will be none.

Materials: Set of beansticks for each child (6 tens sticks and 18 loose beans), set of enlarged beansticks, chalkboard.

FIGURE 3-1a

Sample lesson plan

Name: **Date:**
Subject:

Objective:

Setting the stage:

Instructional input:

Model:

Checking for understanding:

Guided practice:

Independent practice:

Materials:

FIGURE 3-1b
Blank lesson plan form

Unifix cubes, mathematics games and puzzles, measuring devices, books related to mathematics, calculators, listening devices, one or more computers, and other materials. At any given moment, only materials dealing with current work should be accessible to children; the remaining materials are stored on shelves and in cupboards. Rooms without permanent centers can have temporary centers to deal with specific topics.

Classroom learning centers serve several useful purposes. They provide a setting for children to:

learn about a new concept or skill.
practice a previously learned skill.
work in small groups on special projects and enrichment activities.

In a classroom where one or more well-functioning centers operate, children work at centers independently of their teacher. While these children are working independently at centers, the teacher has time to work with individuals and small groups of children whose activities require attention.

A learning center will have one or more stations at which children work. Each station will contain materials needed for one activity or for a group of related activities. When there is more than one station, each station will focus on one aspect of the topic. For example, at a center dealing with linear measure there may be two stations: one that contains centimeter rulers and materials for children to measure to the nearest centimeter; and one with meter sticks so the children can measure longer objects. There may be more than one center in a classroom. If measurement is the focus of study, there may be three centers: one for linear, one for volume, and one for mass (weight) measurement. Children will usually rotate through stations until work at each station is completed. It is advisable to have one more station than there are groups of children so there will be an open station for a group to go to when it is ready to move on.

Work at a station may be directed by problem cards, either commercial or teacher made. The set of cards in Figure 3-2 is designed to provide direction for activities at a learning center set up to study odd and even numbers. The same information could also be put on a worksheet that contains directions and spaces for children's responses. Some teachers record instructions on cassette tapes and have each child listen to directions through a headset. There should be enough cards and sufficient manipulative materials at each station to accommodate individuals or groups of two or three children who work together.

Introducing a New Topic

Activities at a learning center can be used to introduce a new concept or skill. The activities for learning about odd and even numbers on the problem cards in Figure 3-2 can be used to introduce those numbers to children.

(text continues on p. 60)

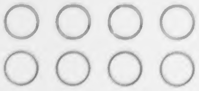
FIGURE 3-2

Problem cards for work with odd and even numbers

ODD AND.EVEN (1)

1. Get a bag of poker chips.

2. Take 8 chips from the bag and put them on your table so they look like this:



Count the chips in each row. There are ___ chips in each row.

3. Use a set of 10 chips this time. Put them in 2 rows with an equal number of chips in each row. Draw a picture of your two rows of chips. There are ___ chips in each row.

(2)

4. Put a set of 6 chips in 2 rows with an equal number of chips in each row. Draw a picture of your two rows of chips. There are ___ chips in each row.

5. Do the same things with each of these sets of chips. Draw a picture of each set of chips after you have put them in two rows. Tell how many chips are in each row by writing the answer on your paper.

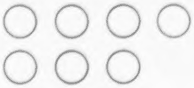
a. 4 chips	d. 20 chips
b. 12 chips	e. 18 chips
c. 16 chips	f. 30 chips

FIGURE 3-2
(continued)

(3)

6. Make some other sets that have equal numbers of chips in each row when you make 2 rows. Draw a picture of each of your sets. Tell how many chips are in each of your sets.

7. Make a set of 7 chips and put them in 2 rows like this:



Do the rows have the same numbers of chips in them? There are ___ chips in one row and ___ chips in the other row.

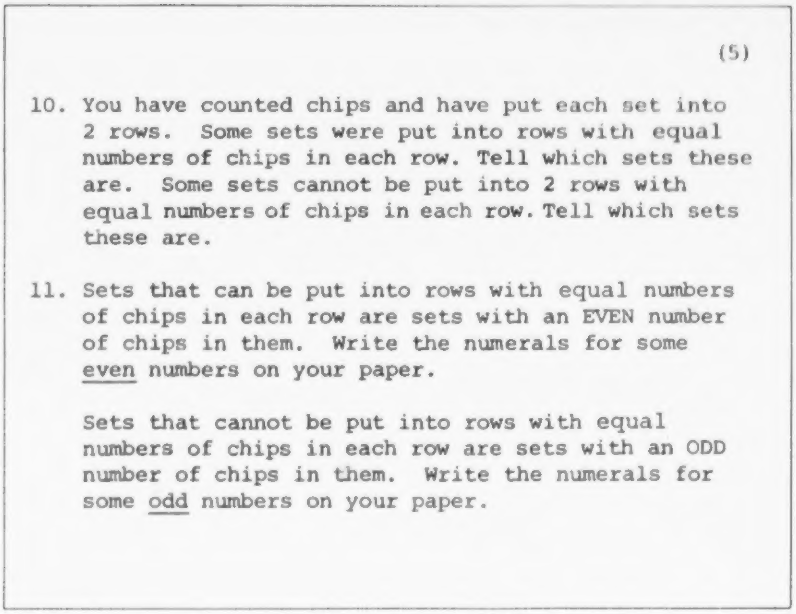
(4)

8. Make 2 rows with 5 chips. Draw a picture of your two rows. Did you put an equal number of chips in each row? There are ___ chips in one row and ___ chips in the other row.

9. Make 2 rows with the following sets of chips. Do it so there is always one more chip in one row than in the other row. Draw a picture of each of your sets after you have put them in 2 rows.
a. 3 chips d. 11 chips
b. 9 chips e. 19 chips
c. 15 chips f. 23 chips

FIGURE 3-2

(continued)

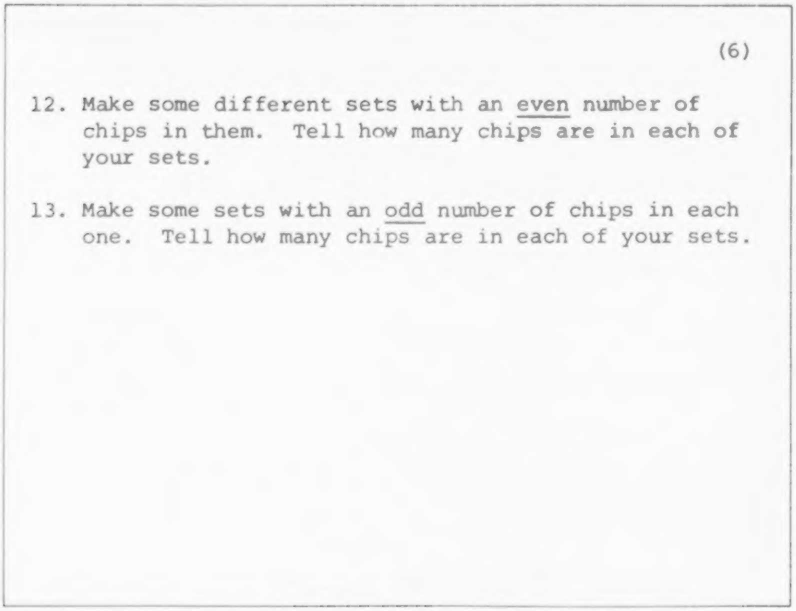


(5)

10. You have counted chips and have put each set into 2 rows. Some sets were put into rows with equal numbers of chips in each row. Tell which sets these are. Some sets cannot be put into 2 rows with equal numbers of chips in each row. Tell which sets these are.

11. Sets that can be put into rows with equal numbers of chips in each row are sets with an EVEN number of chips in them. Write the numerals for some even numbers on your paper.

    Sets that cannot be put into rows with equal numbers of chips in each row are sets with an ODD number of chips in them. Write the numerals for some odd numbers on your paper.



(6)

12. Make some different sets with an even number of chips in them. Tell how many chips are in each of your sets.

13. Make some sets with an odd number of chips in each one. Tell how many chips are in each of your sets.

Activities dealing with diagonals (see Problem 8-1), six-square nets (see Problem 8-2), and probability (see Problem 14-3) are examples of topics that could be put on problem cards or worksheets to introduce a topic.

Practicing Previously Learned Skills

Learning centers provide an ideal setting for practicing previously learned material. Many games, puzzles, tape cassette and computer programs, and other materials provide a variety of approaches for practice with basic fact combinations, algorithms, measuring, probability, problem solving, and other skills. These materials should be related to teacher-directed lessons and made easily accessible in learning centers so children will think of them as an integral part of the mathematics program rather than as something special. Practice materials in learning centers should be changed frequently. It is better to remove an item before children's interest in it has peaked than to leave it until children become bored with it. The item will be received with enthusiasm when it is returned.

Enrichment Activities

Enrichment activities are those that enable some children to investigate a topic in more depth than the class as a whole or to deal with a topic that is not a part of the regular mathematics curriculum. Examples of enrichment topics include historical numeration systems, such as the Mayan or Egyptian systems; alternate algorithms for computing different operations, such as the Russian peasant multiplication on page 377; and the history of measurement. In Chapter 13, a three-level plan that deals with an investigation of specialized measuring instruments is described, and procedures for setting up a center featuring this enrichment plan are explained.

Directing Work at a Learning Center

Center work must be planned with the same care as teacher-directed lessons. You must select one or more topics, determine your children's readiness to cope with them, and decide which level(s) of mathematical behavior will be stressed. In addition to planning and organizing materials and activities, you must take care to orient children to centers. Children with a background of experiences in centers will need less orientation than those working in centers for the first time. First-time children need special attention. Their first days need to be planned carefully, with more attention given to procedures than to mathematics activities. Once children are familiar with procedures, their attention will shift to the activities. Children vary in their abilities to adjust to new learning situations, so it is likely that some will become independent quite soon, while others will need direction from you or another adult for days or even weeks before they are capable of working effectively.

A desirable scheme is one that will enable children to function at a learning center with a minimum of supervision. Children need to know the purposes of activities you assign them. Certainly each child's activities need to be related to his or her needs as revealed by your observations, tests, and other assessment procedures. One way to give children information about activities and purposes is with a contract. The work included in a contract may be selected by a child with your approval or specified by you. It may require a day, a week, or longer to complete. The contract should contain one or more performance objectives to identify what the child will be able to do upon successful completion of the contract; and a description of the activity or activities. The contract in Figure 3-3 governs work at a five-station learning center dealing with addition and subtraction of common fractions. The activities are designed to reinforce and strengthen children's understanding of these operations and would follow teacher-directed lessons that introduced them.

When a contract has been completed, it should be returned to you to be discussed and evaluated. You can meet with children individually or in groups or with the class as a whole. Discussion is important for at least two reasons: (1) Talking about the topic with you and their peers helps children internalize concepts and clear up any misconceptions; (2) You can evaluate children's learning, noting and correcting misunderstandings, and you can collect information for further planning and use in reporting pupil progress, whether by grade on a report card or by a parent-teacher conference. If center work is to continue, a new contract can be drawn up at the conclusion of the discussion. Naturally you will not have such an elaborate contract for every center activity. You may orally instruct children about their work, giving them directions for a given period of time. There will also be times when children need access to materials in a center in order to complete a seat work assignment or project. A center should be open to children who want to spend free time working there.

INDIVIDUALIZED INSTRUCTION

A number of schemes to enable children to pursue individual study may be used. The simplest is to let each child complete the contents of a mathematics textbook at his or her own pace. Computer programs also provide a means for individualizing instruction.

FIGURE 3-3

Sample contract. Sources: Game in Station 3 is from Leonard M. Kennedy and Ruth L. Michon, Games for Individualizing Mathematics Learning *(Columbus, Ohio: Charles E. Merrill, 1973), pp. 88–90. Film in Station 4 is from Jott Films, PO Box 745, Belmont, Cal. 94002. Cassette in Station 5 is from* Mathematics Teaching Tape Program, Primary Level *(Houghton Mifflin Company, 110 Tremont Street, Boston, Mass. 02107).*

Adding and Subtracting Rational Numbers

Name ________
Date to be completed ________
Date returned to teacher ________

Performance Objective: When you have satisfactorily completed the activities at this center, you will be able to add and subtract fractional numbers represented by fractions having like denominators.

________ *Station 1.* There are two envelopes containing materials for activities dealing with addition of fractional numbers. Read the directions on the envelopes, then complete the activities on each problem card. Write your answers on a response card. Return the materials to the right envelopes.

________ *Station 2.* There are two envelopes containing materials for activities dealing with subtracting fractional numbers. Complete the activities and return the materials to the right envelopes. When you have finished all of the activities, staple your response card to your contract.

________ *Station 3.* Choose a partner to play "Common Fraction Addition." The directions are on the cover of the box holding the game. Be sure to put the board and playing pieces back in the box when you are finished.

________ *Station 4.* Watch the film loop *Fractions: Adding and Subtracting (Common Denominators).* You can watch the film more than once, if you wish.

________ *Station 5.* The listening center has a cassette called *Adding Fractional Numbers.* Get one of the work sheets and put your name on it before you listen to the tape. When you have completed the work sheet, staple it to your contract.

Even though both of these schemes permit children to move at their own rates, minimizing the chances that individuals are moving at a pace that is either too slow or too fast, both schemes have weaknesses that limit their effectiveness. One weakness is that a child is denied the opportunity to work with other children. The child does not reap the benefits of sharing ideas about computing, measuring, solving problems, and other topics in group discussion. Neither do individuals have opportunities to work together during problem solving, measurement, geometry, and other group-oriented activities. A major weakness of using a textbook to individualize learning is that a textbook is not designed to be used by a child alone. Some children can read and understand the pages on which new topics are introduced and move on from these pages to the activities that follow. But many children cannot cope with these pages on their own and develop poor understandings of the concepts and processes. Since children reach the same page of a book at different times, teachers who use a textbook to individualize learning face the disadvantage of having to explain concepts and procedures many times. Using a textbook or a computer severely limits a child's access to a variety of materials for building an understanding of mathematics. A major weakness of any individualized program is that over time a child is likely to become bored by the repetitious nature of what she or he is doing each day.

A well-conceived program for individualizing instruction is not easy to plan, develop, and execute. Your plan must provide ways to determine each child's mathematical skills and understandings at entry into the program and at intervals during the year. Once a child's background has been determined, a plan for work must be implemented. A meeting with each individual will enable you and the child to discuss materials and procedures. Contracts that direct work at learning centers, with calculators and computers, and with textbooks can be used to direct the child's work for a specified period of time. A new contract is drawn up as each old one is completed. If you have a teacher aide, you should determine ways for that person to assist children with their contract work. You should not, however, relinquish responsibility to an aide for assessing a child's progress and establishing a program of activities. An aide should help a child only after you have prescribed which materials and procedures are to be used.

Teachers who use learning centers or who individualize instruction must develop a tolerance for ambiguity. Classroom activities will be less structured than when teacher-directed lessons are the norm. Not all children will be doing the same thing at the same time. Noise levels in the classroom will also rise. While sounds should never rise to the point where they disturb others in the room or in adjoining rooms, children need opportunities to move about, to discuss their work with others, and to work with machines and materials that generate sounds. When learning centers and individualized instruction are used, children themselves will often decide when a contract or activity has been completed. You will have the opportunity to consider the completeness of their work when you meet with them to discuss their contract. You also need to allow time for children to investigate and ponder as they work and to suspend judgment on their activities until they declare that their work is done. This does not mean that you will never intervene; it does mean that your intervention will be judicious.

MATHEMATICS LABORATORIES

A mathematics laboratory, as defined in this book, is a special schoolroom in which children learn mathematics. It is equipped with a wide array of commercial aids, such as Cuisenaire rods, Unifix cubes, abacuses, geoboards, measuring devices, computers, and calculators; and with teacher- and child-made materials, such as cut-out "fraction pies" and fraction strips. There are also viewing and listening centers for film strips and cassette tape programs. Mathematics games, puzzles, and books are also available. A mathematics teacher-specialist oversees the work, which may be directed by the specialist or an aide. Children may work individually, in small groups, or as a whole class. Both teacher-directed and center activities are a part of the laboratory experience. In some schools all mathematics instruction occurs in a laboratory. Children are assigned by class or by group. A child's place in a group is determined after a thorough diagnosis has been completed. Membership in a group is not permanent but remains flexible to accommodate each child's change in performance. In other schools only children with special needs work in a laboratory. Children with handicaps receive special help for problems that cannot be accommodated in a regular classroom. Gifted and talented children have opportunities to pursue topics of special interest. Teachers from the school and neighboring schools may come to the laboratory for workshops or to make things for their own classrooms.

EQUITY IN MATHEMATICS

Since the 1970s, equity in mathematics has been receiving national attention. Equal opportunity for students of both sexes and all minorities gives everyone the chance to acquire the knowledge and skills needed to face the future with confidence.

A 1972 survey by sociologist Lucy W. Sells showed that while 57 percent of the male students who entered the Berkeley campus of the University of California had four years of high-school mathematics, only 8 percent of the female students had the same background.[2] Sells also reported a 1978 survey of California high-school students showing that 79 percent of the students from Asian ethnic groups and 72 percent of the white students were in a sequence of courses leading to calculus. On the other hand, only 25 percent of the Hispanic students and 20 percent of the black students were in such a sequence of courses. While 40 percent of all students taking mathematics courses were black, they comprised 79 percent of the students in remedial courses without access to even high-school algebra and geometry. Only 5 percent of the black students were enrolled in trigonometry, elementary functions, and calculus.[3]

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[2] Lucy W. Sells, "Mathematics—A Critical Filter," *The Science Teacher*, XXXXV, No. 2 (February 1978), p. 28.

The lack of high-school mathematics blocks access for many students, particularly females and minorities, to majors in science, medicine, engineering, business, and other fields in which mathematics is important. Although more recent surveys show that the number of females with sufficient mathematics to choose any major they wish is increasing,[4] there are still too many females and minorities who are blocked from some majors because of too little mathematics.

The inability to enter college and university major programs is not the only limitation placed on future occupations by a lack of mathematics. It is estimated that background in high-school algebra and geometry alone may make a difference of 25 percent in scores for examinations required to enter many civil-service positions and industrial jobs.[5] In a rapidly changing society, it is unlikely that an individual will remain in the same occupational field for a lifetime. Old jobs are eliminated, and new ones, which often require a knowledge of mathematics, replace them. A good mathematical background is a requirement for persons in most managerial positions. Persons who must make occupational shifts or who aspire to move up in their chosen fields are denied opportunities to do so if they have an insufficient knowledge of mathematics.

One reason for the differences between the amount of mathematics taken by males in white and Asian ethnic groups and the amount taken by females and black, Hispanic, and other minority males has been the widespread belief that mathematics is a male domain, particularly the domain of males in white and Asian ethnic groups. There has been a widespread belief that males have an aptitude for mathematics that females do not possess.[6]

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[3] Lucy W. Sells, "The Mathematics Filter and the Education of Women and Minorities," in *Women and the Mathematical Mystique*, Lynn H. Fox, Linda Brody, and Dianne Tobin, eds. (Baltimore: Johns Hopkins University Press, 1980), p. 68.

[4] See Jane M. Armstrong, "Achievement and Participation of Women in Mathematics: Results of Two National Surveys," *Journal of Research in Mathematics Education*, XII, No. 5 (November 1981), pp. 365–366; John Ernest, "Mathematics and Sex," *American Mathematical Monthly*, LXXXIV, No. 8 (October 1976), pp. 601–602; Sheila Tobias, "Managing Math Anxiety: A New Look at an Old Problem," *Children Today*, VII, No. 5 (September/October 1978), p. 8.

[5] Sheila Tobias, "Managing Math Anxiety: A New Look at an Old Problem," p. 7.

[6] Elizabeth Fennema, "Women and Girls in Mathematics—Equity in Mathematics Education," *Educational Studies in Mathematics*, X, No. 4 (November 1979), p. 389.

Support for this belief came from the fact that males outnumber females in mathematics-oriented careers by large numbers, and that males as a group have higher scores than females as a group on tests such as the mathematics portion of the *Scholastic Aptitude Test* and the high-school portion of the *National Assessment of Educational Progress* mathematics tests. Elizabeth Fennema and Julia Sherman conducted research showing that the conclusions drawn from comparisons of scores on such tests are erroneous because they do not take into account the amount of mathematics that students have had before they take the tests.[7] Fennema summarizes her research by saying:

> (1) No sex-related differences are evident—at any cognitive level, from computation to problem solving—at the elementary level (this conclusion has been accepted for a number of years); (2) after elementary school, differences do not always appear; (3) starting at about the seventh grade, if differences appear, they tend to be in the males' favor, particularly on tasks involving mathematical reasoning; (4) there is some evidence that sex-related differences in mathematics learning in high school may not be as great in 1978 as they were in previous years; and (5) conclusions reached about male superiority have often been gathered from old studies in which the number of mathematics courses taken was not controlled. Therefore, males with more mathematics background were being compared with females with less mathematics background. In reality, then, the comparison was not between females and males but between students who had studied mathematics from one to three years in high school and students who had studied mathematics for two to four years of high school.[8]

Erroneous beliefs about the superiority of one group of students over another group in mathematics has led to differential treatment between the two groups. Teachers who believe that females are less capable than males often treat females differently. There is little, if any, difference in treatment in the primary grades. As children advance, however, teachers frequently interact more with males, paying more attention to their problems, giving them more help and praise, and expecting a higher level of achievement. Males are encouraged to become independent as they work, while females are encouraged to become more dependent.[9]

Students for whom there is a lower level of expectation will frequently not achieve as much in mathematics as students for whom expectations are higher. You should not let stereotypical beliefs about the capabilities of female or minority students in mathematics cause you to treat them differently or with different expectations than you have for male students in white and Asian ethnic groups.

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[7] Elizabeth Fennema and Julia Sherman, "Sex-Related Differences in Mathematics Achievement, Spatial Visualizations, and Affective Factors," *American Education Research Journal*, XIV, No. 1 (Winter 1977), pp. 51–71.

[8] Elizabeth Fennema, "Sex-Related Differences in Mathematics: Where and Why," in *Women and the Mathematical Mystique*, Lynn H. Fox, Linda Brody, and Dianne Tobin, eds. (Baltimore: Johns Hopkins University Press, 1980), p. 81. Used by permission.

[9] Elizabeth Fennema, "Teachers and Sex Bias in Mathematics," *Mathematics Teacher*, LXXIII, No. 3 (March 1980), pp. 169–170.

MATHEMATICS ANXIETY

> "My first awareness that I couldn't do mathematics came in the sixth grade when I found that I could never figure out how to do word problems. Later I had difficulty doing division with fractions and working percent problems. By the time I was in the seventh grade, I dreaded going to math class."

> "The problem got so bad in the ninth grade that it seemed like a wall arose between my eyes and my mind every time I faced anything to do with math. Nothing in class or the book made any sense at all. Today I avoid situations that involve numbers. Why, I even walk rather than take a cab because I don't have to figure a tip if I walk."

These statements are typical of those made by adults who are victims of mathematics anxiety. Mathematics anxiety is a fear of mathematics or an intense, negative emotional reaction to the subject. Mathophobia is a serious problem for many people. Some researchers who have studied the problem contend that a majority of adults suffer from mathematics anxiety to some degree.[10] Even though the problem does not usually manifest itself until after the elementary school years, evidence shows that it frequently gets started there. For students, it may eventually evoke such strong feelings that they go to great extremes to avoid taking mathematics courses or courses that require a mathematics background.

Adults who suffer from mathematics anxiety sometimes attend therapy workshops designed to help themselves overcome their anxiety. Almost without exception they report that certain teacher practices and expectations contributed to their anxieties. Five such practices are:

- Emphasis on memorization
- Emphasis on speed
- Authoritarian teaching
- Emphasis on doing one's own work
- Lack of variety in teaching-learning processes

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[10] See Mitchell Lazarus, "Mathophobia: Some Personal Speculations," *The National Elementary Principal*, LIII, No. 2 (January/February 1974), pp. 15–22; and Sheila Tobias, *Overcoming Math Anxiety* (New York: W. W. Norton, 1978).

a. Emphasis on Memorization Most children can and should be expected to learn the basic facts for the four operations with whole numbers and learn an algorithm for each operation. However, when they are expected to learn these facts and processes by memorization alone and in a manner devoid of meaning, many children experience learning problems. Rather than seeing mathematics as a structured body of knowledge, they learn a disconnected collection of facts and rules. Children usually keep up in the early grades where there are few facts and rules, although even then there are some who lag behind. By the time they reach the fifth and sixth grades, many children are overwhelmed by all they are expected to memorize. Assignments are a chore to finish. They dread having to take tests. They cannot apply what they have learned, because they see little or no connection between mathematics and the real world. Many of these children are filled with anxiety by the time they reach junior high school and vow to quit mathematics as soon as possible.

b. Emphasis on Speed Memorization and speed usually go hand in hand. Teachers use drills, timed tests, and relays and games that put a premium on speed to foster memorization. Some children can quickly recall memorized facts and processes and are not affected by these tactics. Others, however, work more deliberately. When these children are forced to work at uncomfortable speeds, they often become apprehensive about their ability to cope with mathematics. Deliberate and persistent workers should be given time to complete their assignments without pressure. They also need praise and credit equal to that given to students who finish more quickly.

c. Authoritarian Teaching Persons in mathematics anxiety workshops frequently recall how their teachers gave them a step-by-step procedure, along with a list of rules, for doing long division or other procedures. Teachers often stated that theirs was the only acceptable way to do the work and that other methods would be viewed with disfavor. This is authoritarian teaching of mathematics. Authoritarian teachers also emphasize correct answers and give no credit for children's processes. When emphasis is placed on a "right way" of working along with right answers, children often think of mathematics as being inflexible and lacking in creativity.[11]

d. Emphasis on Doing One's Own Work Teachers commonly expect children to work alone in mathematics. From the primary grades on, children are admonished to "Do your own work." Children are often forbidden to seek help from a classmate. Sheila Tobias says that teachers should provide opportunities for children to do mathematics in small groups, so they can work together on problem-solving strategies and processes and compare answers.[12] There are many times, particularly during the concrete-manipulative stage of the teaching-learning process, when children can be formed in small groups to work on a new topic.

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[11] Wade H. Sherard, "Math Anxiety in the Classroom," *The Clearing House*, LV, No. 3 (November 1981), p. 106.

e. Lack of Variety in the Teaching-Learning Process Lorelei Brush reported in her study of New England students in grades six to twelve that one of the major reasons given for disliking mathematics was lack of variety and boredom.[13]

> . . . These students enjoy being presented with problems which make them think, they derive satisfaction from satisfactorily answering a tough question, and they do not enjoy repetitive assignments which require them to solve a series of very similar problems.[14]

The common practice of centering mathematics instruction around a textbook, speed drills, and timed tests is uninspiring and repetitive and contributes to many children's dislike of mathematics. The type of teaching advocated in this book provides you with a repertoire of skills and procedures that will provide the variety you need to help children maintain their interest as they learn mathematics.

LESSENING MATHEMATICS ANXIETY

Mathematics is so important for today's students that anxiety must not be allowed to cause any student to discontinue study of the subject. Teachers cannot let their attitudes and practices contribute to mathematics anxiety. The practices that follow are ones that are helpful for reducing stress in the classroom, improving students' attitudes, and encouraging them to continue their study of mathematics.

> Provide a relaxed, unhurried atmosphere within which children work without pressure. Assignments should not be excessive for the in- and out-of-school time children have to devote to mathematics. Children should know that they will complete each assignment in a reasonable time if they stay on task.

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[12] Sheila Tobias, "Stress in the Math Classroom," *Learning*, IX, No. 6 (January 1981), p. 38.

[13] Lorelei Brush, *Encouraging Girls in Mathematics* (Cambridge, Mass.: Abt Books, 1980), p. 49.

[14] Lorelei Brush, p. 50.

Help children be aware of the usefulness of the mathematics they are learning. Help them see ways mathematics is applied to the solution of both everyday and nonroutine problems. Provide information about occupations that require a mathematics background. Careers as diverse as agriculture, business, elementary education, food science, geography, and premedicine require a high level of mathematical understanding. Students must keep their options open, because many careers are requiring an increasing amount of mathematics. A list of addresses in Appendix A identifies sources of information about careers that require advanced mathematics and about other topics related to anxiety in mathematics.

Give credit for processes as well as correct answers. Do not insist that work be done "the teacher's" way. Lorelei Brush reports that one of the primary reasons for students' dislike of mathematics is that, as the years pass, they spend an increasing amount of time learning what is already known and less time contributing their own ideas. She recommends that children be given opportunities to figure out for themselves, alone or in small groups, more of what they learn. Open discussions give students opportunities to express their ideas about and to internalize what they are learning.[15]

Avoid insensitive teaching behaviors, such as humiliation and ridicule. Children's incorrect responses are usually the result of an incomplete understanding of a concept or procedure. Children who are "put down" for incorrect responses become reluctant to speak out, and teachers lose opportunities to analyze and correct faulty thinking.

Do not let yourself get in a rut. Use materials and teaching-learning processes like the ones described in this book. Become acquainted with other mathematics resource books and journals. The *Arithmetic Teacher* contains descriptions of many worthwhile materials and procedures. It is a must for every conscientious teacher of elementary-school mathematics.

Do not overemphasize one aspect of mathematics, such as computation, at the expense of other areas.

Do not use mathematics as punishment. Children view with distaste work that is assigned as a penalty for misbehavior or other classroom infractions.

Maintain a positive attitude toward mathematics. If you are a mathematically anxious person, do not be afraid to admit it. You are not alone! Also, *do something about it!* Books and articles contained in this chapter's bibliography will help.

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[15] Lorelei R. Brush, "Some Thoughts for Teachers on Mathematics Anxiety," *Arithmetic Teacher*, XXIX, No. 4 (December 1981), pp. 38–39.

SPECIAL STUDENTS

The majority of children in a class experience no special problems in learning mathematics when the elements of good teaching are in their units and daily work. All children benefit from teaching that emphasizes the diagnostic-prescriptive approach to learning that is advocated in this book. Special provisions, however, must occasionally be made for two groups of children with special needs. These are students with learning handicaps and gifted-talented students.

Learning-handicapped Children

The Education for All Handicapped Children Act (PL 94-142) requires that all handicapped children between the ages of three and eleven have the opportunity for free public education in what is called "the least restrictive environment." This means that children who formerly may have been taught in special schools or classes are now in regular classrooms for all or part of their education.

Learning-handicapped children are a heterogeneous lot, just as are children in any large group. Even so, there are categories of exceptionality that help differentiate children with one sort of handicap from those with another type. Four categories of handicaps are discussed here:

Learning disabilities
Sensory handicaps
Physical impairments
Social-emotional impairments

(Children handicapped because of mental retardation are not discussed in this book.)

a. Learning Disabilities Some children have learning disabilities that prevent them from learning in the same ways as most children. For example, there may be a visual discrimination deficit that causes a child to confuse similar numerals, such as "3" and "5," and signs of operation, such as "+" and "×." Nancy Bley and Carol Thornton developed a table (Table 3-1) to use as an aid in identifying learning-disabled children.

The table names three types of learning disabilities: perceptual, memory, and integrative. Several subcategories for each disability are listed. Examples of some of the behaviors manifested by children who possess a given disability are also listed. Many of the behaviors are characteristic of children commonly considered to be lazy, inattentive, daydreamers, or

(text continues on p. 74)

TABLE 3-1
Examples of learning disabilities affecting performance in mathematics*

	Visual Deficit	Auditory Deficit
Perceptual		
Figure-ground	May not finish all problems on page Frequently loses place Difficulty seeing subtraction within a division problem Difficulty reading multidigit number (see closure)	Trouble hearing pattern in counting Difficulty attending in the classroom
Discrimination	Difficulty differentiating coins Difficulty differentiating between or writing numbers (3 for 8; 2 for 5) Cannot discriminate between operation symbols Cannot discriminate between size of hands on clock Difficulty associating operation sign with problem (see abstract reasoning)	Cannot distinguish between 30 and 13 (see receptive language) Difficulty with decimal numbers
Reversal	Reverses digits in a number (may also be a sequential memory problem) Difficulty with regrouping	
Spatial	Trouble writing on lined paper Difficulty with concept of before/after, so trouble telling time Trouble noticing size differences in shapes Trouble with fraction concept due to inability to note equal-sized parts Difficulty writing decimals Difficulty aligning numbers	Difficulty following directions using ordinal numbers

TABLE 3-1 *(continued)*

	Visual Deficit	Auditory Deficit
Perceptual Spatial	Difficulty with ordinal numbers Difficulty writing fractional numbers (may also be reversed)	
Memory		
Short-term	Trouble retaining newly presented material	Difficulty with oral drills
	Difficulty copying problems from the board (may be spatial)	Difficulty with dictated assignments
Long-term	Inability to retain basic facts or processes over a long period Difficulty solving multi-operation computation	
Sequential	Difficulty telling time Difficulty following through a multiplication problem Difficulty following through long division problems Difficulty solving column addition problems Difficulty solving multistep word problems	Cannot retain story problem that is dictated
Integrative		
Closure	Difficulty visualizing groups Difficulty reading multidigit number (see figure-ground) Difficulty with missing addends and missing factors Inability to draw conclusions, therefore trouble noticing and continuing patterns	Difficulty counting on from within a sequence

TABLE 3-1 *(continued)*

	Visual Deficit	Auditory Deficit
Integrative Closure	Difficulty with word problems Trouble continuing counting pattern from within a sequence	
Expressive language	Rapid oral drills very difficult	Difficulty counting on Difficulty explaining why a problem is solved as it is
Receptive language	Difficulty relating word to meaning (may be spatial) Difficulty with words that have multiple meanings	Difficulty relating word to meaning Difficulty writing numbers from dictation
Abstract reasoning	Inability to solve word problems Inability to compare size of numbers, using symbols Cannot understand patterning in counting Difficulty with decimal concept	

* Reprinted from *Teaching Mathematics to the Learning Disabled* by Nancy S. Bley and Carol A. Thornton by permission of Aspen Systems Corporation, © 1981.

disinterested. Rather than recognizing that a learning disability may exist, teachers frequently reprimand children who exhibit these behaviors and admonish them to pay attention, try harder, or otherwise shape up. Rather than reprimanding a child who is not learning at the same rate as the other students, you should consider the possibility that one or more learning disabilities exist. At the present time there are no tests or other devices for pinpointing disabilities. However, being aware that disabilities do exist and knowing ways they are manifested in a child's behavior increases the likelihood that you will recognize any learning-disabled children in your classroom. Observations of children during discussions and as they work individually or in small groups, and careful examinations of their written work, will provide clues.

b. Sensory Handicaps Two sensory handicaps that affect children's ability to learn mathematics are visual and auditory (hearing). Naturally there are variations in the severity of sensory losses among children. Children with minor visual deficiencies can usually have prescription glasses, and as long as the glasses are properly fitted and are worn as prescribed, visual losses are no problem. Visually impaired children require large-print materials or special devices in order to read mathematics books and other printed material. Educationally blind children must use Braille material.

Visually impaired and blind children usually adapt easily to the classroom. Their capabilities for learning mathematics are equal to those of their normally sighted peers.[16] Because they often have difficulty in putting ideas together to form concepts, you must present well-sequenced concrete learning experiences during which you help these children organize their thoughts about what they feel and hear, so they can form generalizations about the topics they are studying.

Three factors determine the severity of a child's hearing loss: the age at which the loss occurred, the severity of pitch and frequency losses, and the degree to which the loss has affected language development.[17] A child born with a total hearing loss will have much more difficulty developing and using language than one whose loss is less severe or whose date of onset was after language development had begun. The degree of loss affects the manner in which the child is able to adjust to the classroom. For some children you will need to make only minor or no adjustments in the ways you teach, but a special mathematics program will be needed for others.

c. Physical Impairments There are great variations in the physical impairments that affect children's ability to function in the classroom. They range from health problems such as asthma and diabetes to nervous system disorders like cerebral palsy and epilepsy. Among children with the same impairment, the degree of severity also varies. Since each case is unique, each child's mathematics program will be unique too. Many children require no adjustments, whereas those whose conditions prevent them from writing or manipulating teaching-learning materials require special learning aids and ways of responding to assignments. Specially adapted computer setups have touch or light panels that allow students to respond to the computer's program. Most of the teaching-learning materials used with nonhandicapped children are suitable for many physically impaired children. When the severity of the handicap is great enough to affect future vocational plans and social competency, major adjustments under the direction of specialists are needed.

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[16] Howard H. Spicker and James McLeskey, "Exceptional Children in Changing Times," in Vincent J. Glennon, ed., *The Mathematical Education of Exceptional Children and Youth, An Interdisciplinary Approach* (Reston, Va.: National Council of Teachers of Mathematics, 1981), p. 15.

[17] Spicker and McLeskey, p. 16.

d. Emotional Handicaps Children with emotional handicaps are defined in Public Law 94-142 as being those with a history of having their school performance affected by one or more of these conditions: (1) inability to build and maintain satisfactory interpersonal relations with fellow students or teacher, (2) inappropriate behavior or feelings under normal classroom conditions, (3) deep moods of unhappiness or depression, and (4) a tendency to develop physical symptoms to fears. The definition also includes schizophrenic and autistic children.[18] J. F. Weaver and William Morse broaden the definition to include socially maladjusted children, who make up a large part of the segment of children who have problems in school. They use the expression "socially-emotionally impaired" (SEI) when discussing these children.[19]

According to Weaver and Morse, teachers need to have empathy for socially-emotionally impaired children and be able to accept them, be able to structure the classroom social climate so that SEI children are supported by their classmates, and plan mathematics activities to fit each child.

e. Planning for Handicapped Children A teacher does not work alone when special plans for handicapped children are developed. PL 94-142 prescribes that an individualized educational plan (IEP) be developed for each child. The plan is prepared by a team composed of the child's parent(s), teacher, a person other than the teacher who is qualified to provide or supervise the program, and the child when appropriate. Possibly other persons—school psychologist, special education teacher, or mathematics specialist—may be on the team. Each person provides input for the plan's development, with the overall plan reflecting the group's considered judgment about what is best for the student.

These generalizations should be considered as a plan is developed:

- All handicapped children progress through Piaget's stages of mental development. The rate at which they progress and the level of attainment varies according to the type and degree of handicap. Piagetian-type tests and diagnostic tests are useful for determining the level of mental development and the mathematical skills and understandings of handicapped children.

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[18] J. F. Weaver and William C. Morse, "Teaching Mathematics to Socially and Emotionally Impaired Pupils," in Vincent J. Glennon, ed., *The Mathematics Education of Exceptional Children and Youth, an Interdisciplinary Approach* (Reston, Va.: National Council of Teachers of Mathematics, 1981), p. 96.

[19] Weaver and Morse, pp. 96–97.

The sequence of subtasks for a topic must be considered when selecting content. Gaps in a child's knowledge and understanding are often more critical when handicaps are present because the handicaps make catching up and bridging gaps difficult and time-consuming.

Appropriate learning objectives based on criterion-referenced diagnostic tests should be established for each child.

The teaching-learning sequence—concrete-manipulative, pictorial, representational, and abstract—should be followed with most handicapped children. These children are capable of learning concepts and solving problems, so rote learning and memorization are no more appropriate as modes of learning for them than for other children.

Gifted and Talented Children

No single criterion can be used to identify a gifted student. Formerly a score in the top 1 percent on a standardized intelligence test was a commonly used criterion. Now it is recognized that giftedness manifests itself in many ways. H. Laurence Ridge and Joseph Renzulli identify three clusters of traits that are possessed by individuals at the upper end of the creativity-productivity scale:

- Above-average general ability
- Task commitment
- Creativity[20]

Scores on general intelligence tests are one source of information to use when identifying gifted and talented children. However, when scores alone are used, children with potential for high levels of achievement are omitted. Students who immerse themselves in a particular area of interest (task commitment) and who show originality of thinking and diversity in approaches and inventive solutions to problems (originality) may not be identified by an IQ test; but they possess qualities of giftedness that warrant consideration when the gifted are identified. Your awareness of traits for each cluster will help you identify potentially gifted and talented children. The following are some common traits:

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[20] H. Laurence Ridge and Joseph S. Renzulli, "Teaching Mathematics to the Talented and Gifted," in Vincent J. Glennon, ed., *The Mathematical Education of Exceptional Children and Youth, an Interdisciplinary Approach* (Reston, Va.: National Council of Teachers of Mathematics, 1981), pp. 196–201.

Above-average general ability
accelerated pace of learning—earlier and faster
sees relationships—readily grasps "big ideas"
higher levels of thinking—applications and analysis easily accomplished
verbal fluency—large vocabulary, expresses self well verbally and in writing
extraordinary amount of information
intuition–easily leaps from problem to solution
tolerates ambiguity
achievement and potential have close fit
masters advanced concepts in field of interest

Task commitment
highly motivated
self-directed
accepts challenges, may be highly competitive
extended attention to one area of interest
reads avidly in area of interest
relates every topic to own area of interest
industrious and persevering
high energy and enthusiasm

Creativity
curious
imaginative and perceptive
questions standard procedures
original approaches and solutions
flexible
risk taker and independent

Gifted and talented children are sometimes placed in a special program in their own or another school. If a special program is not available, you will provide for their needs in your class. Although no area of the curriculum should be neglected, gifted and talented children do need frequent opportunities to work in self-selected areas of special interest.

Not all gifted and talented children are equally interested in mathematics, so you cannot expect the same level of accomplishment from each one. However, you should expect that all gifted and talented children will learn basic concepts, skillfully compute with both whole and fractional numbers, use common instruments of measure, and apply mathematics in both in- and out-of-school situations. They will not learn basic concepts and skills by chance; they must participate in a planned, systematic program to develop them. Experiences at the concrete-manipulative, pictorial, and representational levels of learning are important for these children but will often be briefer than for others, as will the time they need to practice facts and algorithms.

In addition to learning basic concepts and skills, gifted and talented children need other mathematical experiences. Two ways of providing additional mathematical experiences are commonly used. Horizontal enrichment provides opportunities for children to explore a variety of topics that are not a part of the usual program. Historical numeration systems and computational processes, nondecimal numeration systems, puzzles, and unusual problems are examples of topics that are commonly used for horizontal enrichment. When vertical enrichment is used, gifted and talented children speed up their study of the regular topics and move on to work that commonly appears in later grades.

Ridge and Renzulli recommend a program that provides three types of activities. First there are general exploratory activities to stimulate interest in specific subject areas. Classroom centers; visits to museums, computer centers, and science laboratories; and talks by pilots, computer programmers, and scientists are examples of first-level activities. Second-level activities build on the first. After choosing an area of interest through first-level activities, the child receives individual or group training to develop processes that are needed to pursue the topic. Library-research skills, facility with special instruments of measure, and knowledge of ways to record special data are developed in phase two. At the third level, the student undertakes an individual or small-group study of a topic. When the study is completed, the student(s) may share the results with classmates through an oral or written report, a classroom or school display, a school assembly, or other form of presentation.[21] An example of how this plan can be used in the area of measurement is described in Chapter 13.

Summary

Four ways of presenting mathematics to children include teacher-directed lessons, learning centers, individualized instruction, and a mathematics laboratory. Teacher-directed lessons are frequently used to introduce new concepts and skills to children. An effective lesson will include all or some of seven elements: (1) setting the stage, (2) statement of objectives, (3) instructional input, (4) modeling, (5) checking for understanding, (6) guided practice, and (7) independent practice. A sample lesson plan that contains notes about each element serves as a guide for teaching a lesson.

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[21] Ridge and Renzulli, pp. 218–222.

Learning centers provide another way of presenting new concepts and skills to children. They are also practical ways to provide practice activities and enrichment experiences for children. A learning center contains materials and instructions for dealing with a single topic. Instructions are provided by problem cards, instruction sheets, recordings, and computer programs. Contracts are sometimes used to direct children's activities at a learning center. Individualized instruction accommodates children's varying styles and rates of learning. While textbooks and computer programs are sometimes used to individualize learning, both materials have weaknesses: Children work alone and do not share ideas or opportunities to work together; children do not have experiences with manipulative materials; the sameness of each day's work frequently leads to boredom; textbooks were not designed for children to use alone; and teachers frequently need to explain concepts repeatedly as children move through their book. Individualized learning programs must be carefully developed in order to be successful. Information about each student's knowledge and understanding of mathematics forms the basis for developing and carrying out a plan that provides activities with a variety of appropriate materials. Some schools provide a well-equipped laboratory as the scene for mathematics instruction. A specialist-teacher provides instruction for all or selected groups of children.

Teachers must assure that all individuals have equal opportunities to learn mathematics. Past erroneous beliefs about the capabilities of women and minority students to learn mathematics have caused them to receive differential treatment that has led them to take fewer advanced courses than males in white and Asian ethnic groups, shutting themselves out of many university majors and professions.

Mathematics anxiety is a fear of mathematics created by conditions inside and outside of school that cause a person to avoid mathematics to some degree. Mild anxiety causes people to feel uncomfortable when they take a mathematics test or when other people observe them compute the cost of a tip. Severe anxiety causes students to discontinue their study of mathematics as quickly as possible. Students who avoid mathematics in high school are blocked from admission to many universities or from majors that require a mathematics background. Teachers sometimes transmit their own dislike for mathematics to children. Classroom practices that emphasize memorization and speed, authoritarian teaching, and a lack of variety in mathematics activities contribute to mathematics anxiety. Teachers should provide a relaxed atmosphere, help children understand what they learn, use a variety of procedures and materials, accept alternate solutions, help children to understand the usefulness of mathematics, and use other practices to lessen anxiety.

Some students—learning-handicapped students and gifted-talented children—need special attention when their particular needs cannot be met by the regular program. A learning-handicapped child must have a special individualized education plan developed by a team of people acquainted with the child. Learning handicaps include learning disabilities, sensory impairments, physical impairments, and emotional maladjustments. Many materials and procedures used with nonhandicapped children can be used with handicapped children. The nature of some handicaps requires that major adjustments be made for children who possess them. Materials and procedures must be adapted to meet their very special needs. Gifted and talented children have above-average general ability, task commitment, and creativity. They need programs that provide for their special abilities and needs.

Study Questions and Activities

1. Examine the teacher's manual for a recent elementary mathematics textbook. Compare several teaching plans given in the manual to the seven-step plan suggested in this chapter. How are the textbook lessons alike or different from the seven steps? Is the lesson in the book oriented toward use of materials, or does it emphasize paper-and-pencil work at the abstract level?
2. Interview a teacher who uses learning centers. Ask about issues such as time management, number of activities, type of activities, grouping of children, and evaluation. What are the advantages, disadvantages, and problems with learning centers according to the teacher?
3. According to Kogelman and Warren, each of the following statements is a myth about mathematics that contributes to mathematics anxiety. Read all the myths and place them in order from most believable to least believable. Compare your order to the order of two or three other students. Which of the myths are the most detrimental to people's attitudes toward mathematics?

ONE: Men are better in math than women.
TWO: Math requires logic, not intuition.
THREE: You must always know how you got your answer.
FOUR: Math is not creative.
FIVE: There is a best way to do a math problem.
SIX: It's always important to get the answer exactly right.
SEVEN: It's bad to count on your fingers.
EIGHT: Mathematicians do problems quickly, in their heads.
NINE: Math requires a good memory.

TEN: Math is done by working intensely until the problem is solved.

ELEVEN: Some people have a "math mind" and some don't.

TWELVE: There is a magic key to doing math.[22]

4. A series of Mathematics Equity Conferences was conducted by the National Council of Teachers of Mathematics in 1982–1983. Participants were encouraged to survey children in local schools to determine prevailing conditions pertaining to equity for females and minorities in their communities. This Sentence Completion Task was one source of data from children.

SENTENCE COMPLETION TASK

Name *Age* *Sex* *Date*

1. On weekends I spend my time . . .
2. When I grow up I hope to work in . . .
3. I do not like people who . . .
4. My parents want me to study more . . .
5. I look forward to . . .
6. The subject in school that will help me most when I get a job is . . .
7. When I grow up my parents want me to work as . . .
8. When I was younger . . .

Duplicate the Sentence Completion Task and administer it to at least 20 students in grades 3–6. This can be done in a group situation in most cases, although you may secure oral answers on a one-to-one basis if you wish. Identify each answer sheet by sex, race, or ethnic group. Divide the sheets into groups by male-female or minority-nonminority. Look for patterns to the answers for numbers 2, 4, 6, and 7. If possible, summarize your data in a table and graph. Write one paragraph summarizing your data, identifying any problem you feel needs to be addressed in your school.

5. Another source of information for the conferences was a picture task. Kindergarten through third graders were asked to "Draw a picture of yourself when you are grown up and at work." Have children in your class do this. Then have each write or dictate some statements for you to record with the picture. Indicate sex, race, or ethnic group for each picture. Separate the pictures according to the occupations they indicate; for example, scientist, professional athlete, truck driver, doctor, teacher, nurse, homemaker, and so forth. Write one paragraph summarizing your data and identifying any equity problems you feel need to be addressed in your school.

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[22] Stanley Kogelman and Joseph Warren, *Mind over Math* (New York: Dial Press, 1978), pp. 30–43.

6. Read Brush's article in the December 1981 *Arithmetic Teacher* (see For Further Reading section). Write a statement explaining how you can prevent your students from developing the four attitudes reported to Brush by students in her study.

7. Interview a teacher who has one or more handicapped students mainstreamed for mathematics. What provisions are included in each child's individualized education plan to account for special needs in mathematics? How are these needs met in the classroom? What special provisions are made with materials or instructional approaches for students with handicaps?

8. Martin L. Johnson (see For Further Reading section) makes suggestions for working with gifted students. How do these suggestions compare to the ideas of Ridge and Renzulli discussed in this chapter? Would Johnson's suggestions tend toward horizontal or vertical enrichment?

For Further Reading

Armstrong, Jane M. "Achievement and Participation of Women in Mathematics: Results of Two National Surveys," *Journal of Research in Mathematics Education*, XII, No. 5 (November 1981), pp. 356–372. The results of national surveys by the Women in Mathematics Project group in 1978 and the National Assessment of Educational Progress group in 1977–78 are reported to provide information on mathematics achievement and participation by both males and females. Data from both surveys and generalizations about the findings are reported.

Brush, Lorelei. *Encouraging Girls in Mathematics* (Cambridge, Mass.: Abt Books, 1980). Brush summarizes the findings from her study of students' attitudes toward mathematics and provides information about ways to help girls overcome anxiety and choose mathematics careers.

———. "Some Thoughts for Teachers on Mathematics Anxiety," *Arithmetic Teacher*, XXIX, No. 4 (December 1981), pp. 37–39. Some reasons for not liking mathematics reported by students are summarized. Brush discusses actions teachers can take to prevent mathematics anxiety.

Chandler, Theodore A. "What's Wrong with Success and Praise?" *Arithmetic Teacher*, XXIX, No. 4 (December 1981), pp. 10–12. The article describes the dynamics of success and praise and suggests ways to avoid pitfalls that make praise backfire. Six problems associated with praise are described, as are ways to use it successfully.

Cunningham, Betty. "Individualized Arithmetic Instruction for Fifth and Sixth Graders," *Arithmetic Teacher*, XXV, No. 8 (May 1978), pp. 44–46. Describes one teacher's plan for individualizing children's work. Includes a sample flow chart and samples of multiplication pre- and post-tests.

Danforth, Marion McC. "Aids for Learning Mathematics," *Arithmetic Teacher*, XXVI, No. 4 (December 1978), pp. 26–27. The aids Danforth describes are procedures for helping individuals overcome learning difficulties. Included are such things as giving specific objectives, having a child verbalize thought processes, using error-analysis cards, using analogies, giving directions several ways, using concrete materials, using cues to avoid errors, and removing frustration.

DeRidder, Charleen, and Donald J. Dessart. "Using a Directed Lesson," *Arithmetic Teacher*, XXX, No. 3 (November 1982), pp. 16–17. A five-step sequence for presenting a specific mathematics skill is described. Teacher's activities at each step are explained.

Fennell, Francis (Skip). "Mainstreaming and the Mathematics Classroom," *Arithmetic Teacher*, XXXII, No. 3 (November 1984), pp. 22–27. Ways to assess children with special needs and develop Individualized Education Plans (IEPs) are presented.

Fennema, Elizabeth. "Women and Mathematics: Does Research Matter?" *journal for Research in Mathematics Education*, XII, No. 5 (November 1981), pp. 380–385. Fennema cites a number of myths about abilities and interests in mathematics. These myths have been damaging but are amenable to change. Causes that are due to social factors can be modified, and educators have a responsibility to know the truth about females' abilities and interests in mathematics.

———."Teachers and Sex Bias in Mathematics," *Mathematics Teacher*, LXXIII, No. 3 (March 1980), pp. 169–173. Research evidence supporting the belief that teachers treat female students differently from male students is cited. Five practices that result in greater equity are listed.

———."Women and Girls in Mathematics—Equity in Mathematics Education," *Educational Studies in Mathematics*, X, No. 4 (November 1979), pp. 389–401. This article discusses beliefs about cognitive, affective, and educational variables in the mathematics education of females. The author concludes that there is nothing inherent that keeps females from learning mathematics. The article contains an extensive bibliography.

———, and Julia Sherman. "Sex Related Differences in Mathematics Achievement, Spatial Visualization, and Affective Factors," *American Education Research Journal*, XIV, No. 1 (Winter 1977), pp. 51–71. A comprehensive study of 589 female and 644 male, predominantly white, students in grades 9–12 revealed few sex-related cognitive differences but many attitudinal differences. The study strongly supports the contention that sex differences are the result of sociocultural rather than cognitive forces.

Fox, Lynn H. "Mathematically Able Girls: A Special Challenge," *Arithmetic Teacher*, XXVIII, No. 6 (February 1981), pp. 22–23. Recent research indicates that many able girls elect not to continue their study of mathematics beyond minimum required courses. Fox discusses six practices that will encourage mathematically able girls to continue their studies.

———, Linda Brody, and Dianne Tobin, eds. *Women and the Mathematical Mystique* (Baltimore: Johns Hopkins University Press, 1980). This book contains a collection of articles that deal with women and mathematics.

Hannafin, Michael J. "Effects of Teacher and Student Goal Setting and Evaluations on Mathematics Achievement and Student Attitudes," *The Journal of Educational Research*, LXXIV, No. 5 (May/June 1981), pp. 323–326. The author cites a study showing that children who are given the opportunity to participate in establishing goals in mathematics have a better sense of what they can accomplish than their teachers. Students given opportunities for self-regulation of instruction had greater confidence in their ability to manage portions of their instruction than those who did not.

Hill, Jan. "The Nonsexist Classroom," *Instructor*, LXXXIX, No. 7 (February 1980), pp. 78–80. Discusses ways to lessen sex stereotypes about female and male roles in an elementary classroom. Includes a consciousness-raising attitude scale for teachers prepared by Barbara Samuels.

Hodges, Helene L. B. "Learning Styles: Rx for Mathophobia," *Arithmetic Teacher*, XXX, No. 7 (March 1983), pp. 17–20. Describes three mathophobic children, and discusses four aspects of learning styles teachers should consider to lessen mathematics anxiety in students.

Johnson, Martin L. "Identifying and Teaching Mathematically Gifted Elementary School Children," *Arithmetic Teacher*, XXX, No. 5 (January 1983), pp. 25–26, 55. Provides rationale and guidelines for mathematics programs with the gifted.

Kogelman, Stanley, and Joseph Warren. *Mind Over Math* (New York: Dial Press, 1978). Discusses causes of mathematics anxiety and describes processes for lessening it.

Larson, Carol Novillis. "Organizing for Mathematics Instruction," *Arithmetic Teacher*, XXXI, No. 1 (September 1983), pp. 16–19. Both content and process are included in a model for mathematics instruction.

Lauritzen, Carol. "Using Every Pupil Response in Mathematics Instruction," *Arithmetic Teacher*, XXXIII, No. 4 (December 1985), pp. 46–47. Get every child involved with answer cards.

Lazarus, Mitchell. "Mathophobia: Some Personal Speculations," *The National Elementary Principal*, LIII, No. 2 (January/February 1974), pp. 16–22. Gives a good explanation of mathophobia (mathematics anxiety), some causes, and some effects on ordinary adult life.

Masalski, William J. "Mathematics and the Active Learning Approach," *Arithmetic Teacher*, XXVI, No. 1 (September 1978), pp. 10–12. Active learning, which involves students in physical interaction with materials or the environment, can be promoted through use of organizing centers, open-ended tasks, and the solution of good problems. Defines each means of active learning and gives examples.

Sears, Carol J. "Mathematics for the Learning Disabled Child in the Regular Classroom," *Arithmetic Teacher*, XXXIII, No. 5 (January 1986), pp. 5–11. Sears suggests a variety of planning strategies to engage children who have different kinds of disabilities.

Sells, Lucy. "Mathematics—A Critical Filter," *The Science Teacher*, XXXXV, No. 2 (February 1978), pp. 28–29. This was one of the first articles to call attention to the fact that females are blocked from taking many courses of advanced study by a lack of mathematics courses in high school.

Sherard, Wade H. "Math Anxiety in the Classroom," *The Clearing House*, LV, No. 3 (November 1981), pp. 106–110. The author cites some causes of anxiety and discusses eight practices that help reduce it.

Thornton, Carol A. "Math Centers for Young Learners," *Learning*, VI, No. 1 (August/September 1977), pp. 56–57. Describes the values of math centers and their uses to motivate and reinforce concepts, strategies, and skills. Includes eight pointers for effective use of centers and the distinction between centers and stations.

Thornton, Carol A., and Barbara Wilmot. "Special Learners," *Arithmetic Teacher*, XXXIII, No. 6 (February 1986), pp. 38–41. Describes how manipulatives are useful in all elementary mathematics topics for special learners; suggestions are equally useful for regular students.

Tobias, Sheila. "Managing Math Anxiety: A New Look at an Old Problem," *Children Today*, VII, No. 5 (September/October 1978), pp. 7–9, 36. Discusses mathematics anxiety and describes how clinical intervention can help students and adults overcome it.

———."Math Anxiety: What You Can Do About It," *Today's Education*, LXIX, No. 3 (September/October 1980), pp. 26GS–29GS. Tobias states that elementary school teachers who feel uncomfortable about teaching mathematics owe it to their students to overcome their feelings of anxiety. She discusses some ways teachers can overcome anxieties.

———. *Overcoming Math Anxiety* (New York: W. W. Norton, 1978). Gives reasons cited by clinic participants for their anxieties and contains chapters that discuss ways individuals can lessen mathematics anxiety.

———. "Stress in the Math Classroom," *Learning*, IX, No. 6 (January 1981), pp. 34, 37–38. Tobias discusses four causes of anxiety and presents alternative practices for lessening its incidence.

———, and Carol Weissbrod. "Anxiety and Mathematics: An Update," *Harvard Educational Review*, L, No. 1 (February 1980), pp. 63–70. Updates research on sex-related differences in mathematics and anxiety to point up the link between the two and their impacts on females and minorities. Contains an extensive bibliography.

West, Tommie A. "Using a Textbook Effectively," *Arithmetic Teacher*, XXX, No. 2 (October 1982), pp. 8–9. Steps in planning lessons around a text are explained. Ways to structure lessons, problems to assign, overcoming a text's weaknesses, and ways of correcting homework are discussed.

Williams, David E. "Warm-Ups—Keys to Effective Mathematics Lessons," *Arithmetic Teacher*, XXXII, No. 1 (September 1984), pp. 40–43. Warm-up activities establish the set for a good lesson.

Chapter

4

Calculators and Computers

In Chapter 4, you will read about

1 reasons for using calculators in elementary mathematics

2 criteria for selecting a calculator for the classroom

3 the importance of teaching calculator skills

4 the development of the personal computer and its impact on availability of computing in education

5 types of educational software and guidelines for selecting educational programs

6 the computer language Logo and ways it can be used to develop thinking skills

7 computer awareness activities

Electronic technology has come to the elementary classroom. One need only look at titles of new journals for teachers to grasp the impact that the calculator, computer, and preprogrammed devices are making on instruction and classroom activities. Along with new journals such as *Teaching and Computers*, *Electronic Learning*, *Classroom Computer Learning*, *The Computing Teacher*, *The Logo Exchange*, and the *Journal of Computers in Mathematics and Science Teaching*, the *Arithmetic Teacher*, a journal for teachers of elementary-school mathematics, regularly features articles about computers and calculators and activities for them. Journals for other subjects also contain articles about calculator and computer applications. It is a rare teachers' conference that does not schedule workshops and other sessions concerning these machines.

CALCULATORS IN THE ELEMENTARY SCHOOL

Reasons for Using Calculators

The National Council of Teachers of Mathematics published a set of guidelines for teachers titled *An Agenda for Action: Recommendations for School Mathematics in the 1980s*. This publication recommended that teachers at all levels take full advantage of the power of calculators and computers.[1] "Full power" of the calculator means that children learn to use it as a learning and computing tool in the same way that they learn paper-and-pencil algorithms and mental arithmetic processes. Once the meaning of an operation is understood and one or more algorithms for computing it have been learned, children should also learn to use a calculator for computing that operation. A calculator frees children from routine computation, which is often tedious and time consuming, giving them time to consider problems and how to solve them. Articles in the *Arithmetic Teacher* confirm that children who use calculators generally become better problem solvers than those who do not use them.

In addition to serving as a problem-solving tool, calculators have other uses in the classroom:

> Developing understanding of numbers. A calculator enables children to count by ones, twos, fives, or any other number very quickly. By pressing the $\boxed{1}$ key, the $\boxed{+}$ key, the $\boxed{1}$ key again, and the $\boxed{=}$ key, a child can count by ones to any number. A child who counts to 1000 this way gains insight into both the sequence of

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[1] National Council of Teachers of Mathematics, *An Agenda for Action: Recommendations for School Mathematics in the 1980s* (Reston, Va.: The Council, 1980), p. 1.

numbers and large numbers. Children can also learn about place value with a calculator; they learn the meaning of 6 in the number 468 when they see that to replace the 6 with a 0, they have to subtract 60.

Gain insight into operations. Calculators can be used to develop understanding of operations and show the rationale for their algorithms. Children who begin with 384 and repeatedly subtract 64 to divide 384 by 64 will have a better understanding of the repeated subtraction algorithm for dividing whole numbers. The multiplication algorithm in the margin can be completed in stages on a calculator. By finding 9 × 362, 80 × 362, and 400 × 362, recording the separate products, and then determining their sum, a child gains insight into how the commonly used algorithm works.

$$\begin{array}{r} \times 362 \\ \times 489 \\ \hline \end{array}$$

Extend children's mathematical experiences. The calculator opens the door to many experiences that children might otherwise bypass, because the necessary computations are tedious and time consuming. For example, children can study the pattern that develops as these multiplications are completed:

$$\begin{array}{c} 1 \times 1 \\ 11 \times 11 \\ 111 \times 111 \\ 1111 \times 1111 \end{array}$$

Once these are completed, children can predict the product of 11,111 × 11,111. Does the calculator show that the prediction is accurate? (Some calculators will show only a partial answer but enough to show the estimate's accuracy; others will indicate an overload and no answer.) There are many similar patterns children can study.

Rounding off and estimating. A calculator user should first estimate an answer and then determine if the machine's answer is in the "ball park." An answer that does not reasonably match an estimate should be redone to confirm that it is either accurate or incorrect. Estimated answers are frequently used in problem-solving situations. Rounded numbers are used in most estimations that involve computed answers. Children can use a calculator to learn how to round off numbers. To round off a number, say 74, to the nearest ten, first display the number; then count backward by ones from 74 to 70. Reset the number 74 and count by ones to 80. Because the count to 70 is completed in four steps, and the count to 80 requires six steps, 74 is rounded off to 70. A similar process is used for rounding off to the nearest hundred or thousand.

Work with calculators should not replace knowledge of basic facts and paper-and-pencil computation skills. Even though calculators will produce correct answers for all computations, a child must know which keys to press and the order in which to press them. This knowledge comes from a good understanding of each operation and the algorithms by which it can be processed. Children should not become so reliant on calculators that they use them when simple mental or paper-and-pencil processes are more efficient. A minority of children have difficulty learning facts and computation skills. A calculator may make it possible for these children to learn to compute and to gain a more positive attitude toward the subject.

Studies of calculators in the elementary school indicate that they enhance learning of computational skills and problem-solving processes, or at least are not detrimental. Robert Reys and his associates report that many teachers who have used calculators say that the machines enable children to work more problems and to cover more topics. More than 80 percent of the teachers in Reys' study reported that students were more eager to work and had an enhanced confidence in their abilities to solve problems.[2] This last finding suggests that calculators may be helpful in lessening mathematics anxiety, which, if true, is a further justification for their use.

Criteria for Choosing a Calculator

Do not consider choosing a calculator with less than an 8-digit display. The machine should have a floating decimal. Another thing to consider, at least for children in grades four through six, is whether or not the machine has memory keys. A calculator with 4-key memory stores intermediate answers as multistep computation is completed. Percent and square-root keys are also useful for older children. Size is another factor to consider, especially when machines are selected for very young or physically handicapped children. Very small machines are unsuitable for these children. The Sharp ELSI Mate El-240S is an example of a machine that has the above features; it costs about six dollars. Improvements in battery longevity and solar-powered calculators mean that the machines can be used without concern for recharging them frequently, if ever.

The number of calculators you need depends upon how children will use them. In the primary grades, children may work in learning centers where free investigations for exploring and verifying mathematical ideas and problem-card activities are completed. Only three or four machines are needed. More machines will be needed as they become computational and problem-solving tools. If machines are used during tests, you will need a calculator for each child.

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[2] Robert Reys, *et al.*, "Hand Calculators—What's Happening in Schools Today?" *Arithmetic Teacher*, XXVII, No. 6 (February 1980), p. 41.

Establishing Guidelines for Using Calculators

Despite the availability of calculators and their widespread use among adults and evidence of their usefulness to children, many teachers, administrators, and lay persons object to use of the machines by children in the elementary school. Teachers who choose to have calculators in their classrooms may need to convince administrators, parents, and other teachers that use of calculators by children will not replace other worthwhile arithmetic skills. Rather, a calculator will broaden and extend mathematics concepts and problem-solving activities.

Before you have children begin using calculators, determine if there is a written policy governing their use in your school. If there is, make certain that your children's uses of the machine are consistent with the guidelines in the school's policy. If there is no policy governing use of calculators, you will want to help establish one; and, in the absence of district or school guidelines, you will want to discuss prospective guidelines with your principal. Guidelines should be developed with teacher, parent, and administrator input and should define the limits of use of the machines by children. The authors of this book recommend that once children learn to use a calculator, they should use it extensively in both routine and nonroutine problem-solving situations, including tests and homework. The policy should also describe a procedure for accounting for each machine. A scheme for checking out a machine, whether for use in the classroom or at home, must be established. In some schools, student monitors are appointed to assume the responsibility for checking machines in and out. In others, teachers retain this responsibility for themselves.

Introducing Children to Calculators

Most mathematics textbook series now include calculator activities throughout the grades. Children need to learn about a calculator and the meaning and uses of the different function keys, however, before they can do the textbook activities. A good introductory activity for primary-grade children is making the calculator count up or down by using the equal (=) key. The following activity is a simple one for teaching them to do this:

ACTIVITY 4-1

Calculator Count-Up

Tell each child to press "1", then to press "+", and then to press "1" again.

1 + 1

Tell each child to press the "=" key.

$$\boxed{=}$$

Have a child read the number on the display. (2)
Tell each child to press the "=" key.

$$\boxed{=}$$

Have another child read the number on the display. (3)
Tell each child to press the "=" key again.

$$\boxed{=}$$

Have another child read the new number on the display. (4)
Tell each child to keep pressing the "=" key and to watch the numbers on the display.
Tell the children to stop when the number reaches 50.
(Each calculator should have shown the sequence "2, 3, 4, 5. . . .")

A countdown activity begins at a large number; a child subtracts 1 each time the equal key is pressed. The countdown continues until zero

$$100\ \boxed{-}\ \boxed{1}\ \boxed{=}\ \boxed{=}\ \boxed{=}\ \boxed{=}\ \boxed{=}\ \boxed{=}$$

shows on the display. These counting plans work on most inexpensive calculators, although some require that the +1 or −1 be put into the machine's memory. After you introduce children to the repeated addition and subtraction activities, have each one race up to a given number or down from a given number to see how long it takes to reach a target number.

Older children should investigate the order of operations on a calculator. In the following activity, children learn answers to these questions: "How many different answers are possible when we use parentheses for grouping pairs of numbers in this sentence?" "What is the correct answer when the normal precedence of operation rules is used?"

ACTIVITY 4-2 What Is the Correct Answer?

There are a number of different answers for this sentence, depending upon where parentheses are placed. Write the sentence on the chalkboard.

$$5 \times 3 - 2 + 8 \div 4 = \square$$

Put parentheses around 5 × 3.

$$(5 \times 3) - 2 + 8 \div 4 = \square$$

Tell each child to multiply the 5 times 3, then to complete the other operations in order.

$$\boxed{5}\,\boxed{\times}\,\boxed{3}\,\boxed{-}\,\boxed{2}\,\boxed{+}\,\boxed{8}\,\boxed{/}\,\boxed{4}\,\boxed{=}$$

"What is the answer?" (5.25)

Erase the parentheses from the 5 × 3; put them around the 3 − 2.

$$5 \times (3 - 2) + 8 \div 4 = \square$$

Tell each child to subtract 2 from 3, then to complete the other operations in order. "What is the answer?" (3.25)

Repeat with parentheses around 2 + 8, then 8 ÷ 4. Note and record the answers. (1.25 and 15)

Show the children how parentheses are used to show the normal precedence for doing the operations in the sentence:

$$(5 \times 3) - 2 + (8 \div 4) = \square$$

Have the children note that this answer (15) is the same as one of the answers above. Ask them to tell you why.

Repeat the activity with these sentences.

$$12 \div 4 + 6 \times 3 - 3 = \square$$
$$48 + 6 \div 3 - 4 \times 5 = \square$$

COMPUTERS IN THE ELEMENTARY SCHOOL

The National Council of Teachers of Mathematics recommends that full advantage be taken of the computer in mathematics programs at all levels. The Council states that students should have access to computers with programs and activities that are integrated into the core mathematics program; that a computer literacy course should be a part of every student's general education; and that use of computers in home and schools should be coordinated.[3]

The Personal Computer

Microcomputers are a recent addition to a long line of computing devices. Fingers, stones, and knotted ropes plus the abacus and other devices have been used by people through history to help themselves remember and calculate numbers. The first operational computer, called ENIAC, was developed in 1945. UNIVAC I, the first commercial computer, was produced in 1951. These early computers were very large, unreliable, and, because of vacuum tubes, were so hot they required constant air conditioning.

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[3] National Council of Teachers of Mathematics, *An Agenda for Action: Recommendations for School Mathematics in the 1980s* (Reston, Va.: The Council, 1980), pp. 9–10.

In the 1960s transistors and diodes replaced vacuum tubes, and it became possible to manufacture smaller, more reliable, and less expensive computers. They were still too expensive for most elementary schools, but nearly 200 colleges and universities had computers. Although its first uses were in administration, educators began to recognize the computer's potential as a teaching tool. Patrick Suppes and his associates at Stanford University, and others, did early work with computer-assisted instruction at the precollege level. Students worked in their schools at terminals connected by telephone lines to a computer shared by many users. These early studies indicated that computers could have a positive impact on learning; but the cost of the machines still prevented schools from taking advantage of this new aid to learning.

The breakthrough that made it feasible for schools to purchase computers occurred in the 1970s. The integrated circuit, a fingernail-sized silicon chip that contained all the circuit elements of larger machines, led to smaller, more reliable, and less costly computers. By the late 1970s, technology was creating personal computers that were affordable for many schools, homes, and small businesses.

A personal computing system, including a computer, keyboard, disk drive, monitor, and printer, can now be purchased for as low as one thousand dollars. Figure 4-1 shows a typical elementary-school computer system. These machines have more power and memory than earlier ones that cost many thousands of dollars. Microcomputers can be used for processing words, graphics, sound, and data; retrieving and storing information; synthesizing voices and music; and communicating with other computer users wherever telephone circuits are available. Their low cost puts them within reach of schools at all levels. Their role in society and the role they can play in education make it essential that they be used for instruction and learning in all areas and levels of the elementary school.

Computer Programs

The effectiveness of a computer in a classroom depends to a large extent on the variety and quality of the educational software, or programs, available to a teacher and children. Programs are available for every elementary-school subject area. They are broadly classified as computer-assisted instruction (CAI) programs or computer-managed instruction (CMI) programs. CAI programs are available for drill, tutorials, simulations, and educational games. A comprehensive CMI program includes pretests and

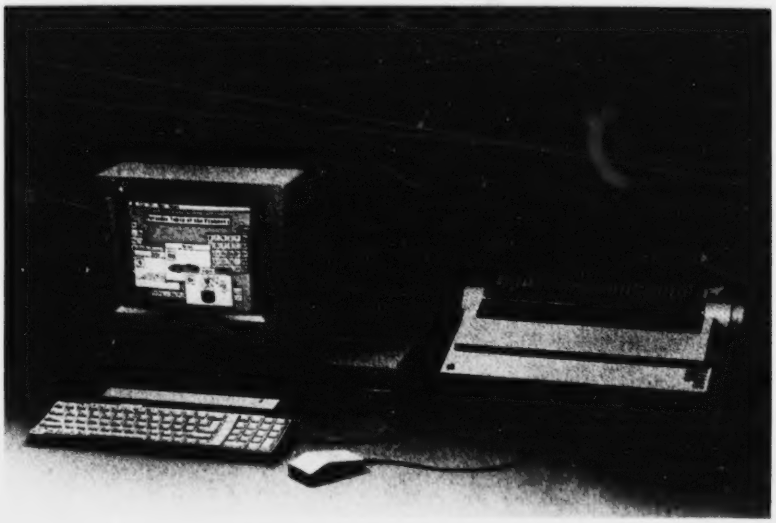

FIGURE 4-1

Typical microcomputer setup, with computer, disk drive, monitor, and printer. Photo courtesy of Apple Computer, Inc., Cupertino, California.

posttests, an independent sequence of activities for each student, and processes for monitoring and recording each child's progress.

The development of a good program for mathematics or any other subject is time consuming and costly. Few persons possess sufficient skills in programming, knowledge of a subject, and expertise in teaching to singlehandedly develop an excellent product. A team composed of a programmer, subject-matter specialist, and educational specialist or instructional designer may spend thousands of hours to develop an educationally sound program.

Software is available from several sources. There are commercial programs developed by for-profit companies, public domain programs, and teacher-produced programs. Programs vary from single-disk, drill-and-practice programs or games to complex management programs on several disks and covering a range of grades. An advantage of commercial programs is that they can be easily used by both teachers and children who are not experienced computer users. A program is usually accompanied by written information about the program, including purpose, intended audience, and directions. Some commercial programs can be modified to make them more appropriate for a particular audience. Public domain programs are available for less cost than commercial programs. These programs are available through networks of computer users who develop and make programs available to others in the network. One such network is the Minnesota Educational Computer Consortium (MECC),[4] which was developed to assist users and educational member systems in coordinating and using computer resources.

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[4] Minnesota Educational Computer Consortium, 2520 Broadway Drive, St. Paul, MN 55113-8097.

Teacher-made programs can be developed to fit the needs of a particular child or group of children. Few teachers possess the programming skills needed, however, for developing their own materials. Although an authoring language such as PILOT (Programmed Instruction Learning or Teaching) has been developed to make programming easy, a great amount of time is needed to produce a worthwhile program.

Learning with Computers

Computer programs provide children with opportunities for drill and practice and for engaging in interactive learning through tutorial and simulation programs. Logo, a sophisticated computing language, provides unusual opportunities for children to do programming and to explore important mathematical ideas at the same time.

a. Drill and Practice Elementary-school children are expected to learn many facts and skills, including knowledge of the basic addition, subtraction, multiplication, and division facts and the ability to compute with one or more algorithms for each operation. A drill-and-practice program can be developed to provide straightforward practice on basic facts and algorithms, or as a complete management system to provide practice on a wide range of skills for several grades. *Math Blaster*[5] is one of the most popular drill-and-practice programs. It provides a choice of any one of the four operations and lets the learner choose the level of difficulty. The drill is presented in a game-type format, in which a child tries to keep a balloon from popping while also answering mathematics problems to increase the score. Many other drill programs have a game format. In *Sum Ducks*[6] a character on the screen tosses rings at ducks in a pond. When the ring goes around the duck's neck, points are added to the score.

An example of a comprehensive drill-and-practice program is *Fact-Master: Addition and Subtraction*.[7] This objective-based program provides instruction and practice with basic facts. The content parallels most elementary-school textbooks and can be used in a classroom or mathematics laboratory to supplement textbooks and other materials. The program includes pretests and posttests for placing children in the program, practice activities, and both individual and class records.

David Moursund reports that drill-and-practice materials are the most commonly used computer programs. He also states that extensive research has proven that computerized drill and practice is an effective aid to learning, but he believes that using a computer for drill alone cannot justify the machine's cost.[8]

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[5] Davidson and Associates, Inc., 3135 Kashiwa Street, Torrance, CA 90505.

[6] Spinnaker Software Corp., One Kendall Square, Cambridge, MA 02139.

[7] Curriculum Associates, 5 Esquire Road, North Billerica, MA 01862.

b. Tutorial A tutorial program differs from a drill-and-practice program by providing primary instruction as well as practice. A tutorial can be designed to provide motivation, state the objectives for learning, give information and examples, guide practice with a few problems to check for understanding, as well as include drill-and-practice activities. The major difference between tutorials and drill-and-practice materials is the provision that tutorials make for providing both informative feedback to the learner and information about the learner's progress as the program is used. Simple tutorials provide the same sequence of activities for every learner. More sophisticated programs respond to the learner and provide individualized feedback based on what the learner knows or is having difficulty with. *FactMaster: Addition and Substraction* also includes a tutorial program with the practice program.

c. Simulations A map is a model, or simulation, of all or part of the world, whereas a terrarium simulates a biological environment. In the domain of computers, a simulation program models a situation in science, economics, art, or some other field to provide a realistic setting in which students deal with alternative situations. Solving problems and understanding relationships are goals in simulation programs. For example, *Sell Bicycles*[9] sets up an environment in which the student plays the role of a bicycle manufacturing company manager who must determine production figures, an advertising budget, and selling prices. The object of the simulation is for the child to produce a profit. *Operation Space Station*[10] deals with plans for a trip to outer space. Students must determine the astronauts, experiments, and supplies needed for a voyage. Many simulations that deal with subjects other than mathematics require that students use mathematics skills. Simulation programs involve students in problem-solving situations for which a variety of strategies are required. (Problem-solving strategies are discussed in Chapter 5.)

Evaluating Software

Before you purchase any program for classroom use, you should evaluate it thoroughly. Guidelines for evaluating software have been developed, using an evaluation form similar to that shown in Figure 4-2. This form deals with

(text continues on p. 100)

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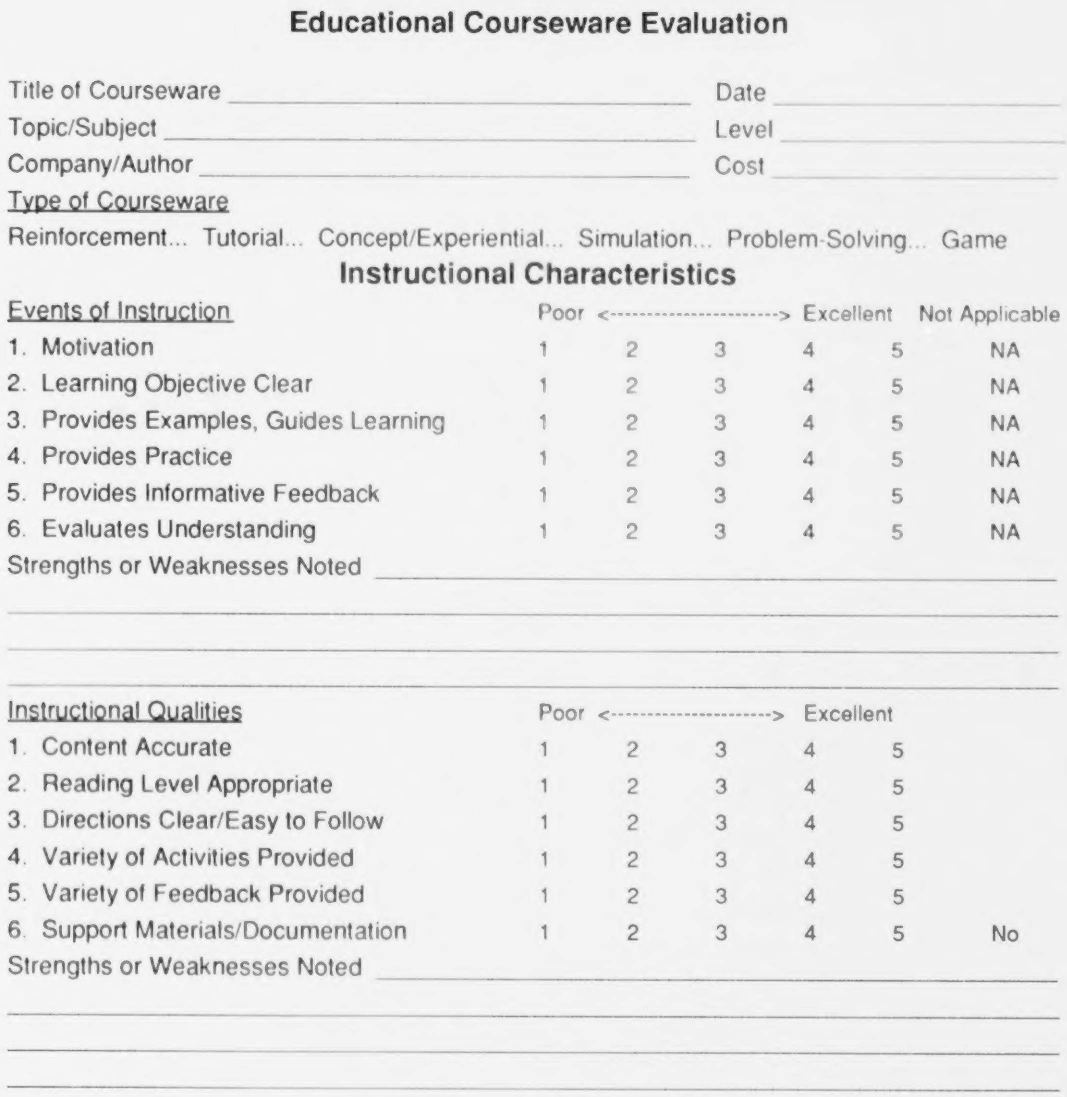
[8] David Moursund, *Introduction to Computers in Education for Elementary and Middle School Teachers* (Eugene, Oreg.: International Council for Computers in Education, University of Oregon, 1981), p. 24.

[9] *Sell Bicycles* was developed by the Minnesota Educational Computer Consortium.

[10] HesWare, 390 Swift Ave, Suite 14, South San Francisco, CA 94080.



Educational Courseware Evaluation

Title of Courseware ______________________ Date ____________

Topic/Subject ______________________ Level ____________

Company/Author ______________________ Cost ____________

Type of Courseware

Reinforcement... Tutorial... Concept/Experiential... Simulation... Problem-Solving... Game

Instructional Characteristics

Events of Instruction	Poor	<----------	---------->	Excellent		Not Applicable
1. Motivation	1	2	3	4	5	NA
2. Learning Objective Clear	1	2	3	4	5	NA
3. Provides Examples, Guides Learning	1	2	3	4	5	NA
4. Provides Practice	1	2	3	4	5	NA
5. Provides Informative Feedback	1	2	3	4	5	NA
6. Evaluates Understanding	1	2	3	4	5	NA

Strengths or Weaknesses Noted ______________________

Instructional Qualities	Poor	<----------	---------->	Excellent		
1. Content Accurate	1	2	3	4	5	
2. Reading Level Appropriate	1	2	3	4	5	
3. Directions Clear/Easy to Follow	1	2	3	4	5	
4. Variety of Activities Provided	1	2	3	4	5	
5. Variety of Feedback Provided	1	2	3	4	5	
6. Support Materials/Documentation	1	2	3	4	5	No

Strengths or Weaknesses Noted ______________________

FIGURE 4-2

Program evaluation form

Technical Characteristics

Effective Use of Media	Poor <---------------------> Excellent					
1. Color	1	2	3	4	5	No
2. Sound	1	2	3	4	5	No
3. Graphics/Animation	1	2	3	4	5	No
4. Speed	1	2	3	4	5	No
5. Page Format	1	2	3	4	5	No
6. Active Participation	1	2	3	4	5	No

Strengths or Weaknesses Noted ____

____

____

____

Implementation	Poor <---------------------> Excellent				
1. Learner Is in Control (determine level, get help, quit)	1	2	3	4	5
2. Program Runs Without Problem (no stalls, bugs, bombs)	1	2	3	4	5

Strengths or Weaknesses Noted ____

____

____

____

Classroom Considerations

	Poor <---------------------> Excellent				
Overall I rate this courseware	1	2	3	4	5

What were the best features of the courseware? ____________________________

____

____

What were the worst features? ____

____

____

Could the courseware be incorporated in existing curriculum and instruction? __________

____

____

Does the courseware add something unique, or not presently available, to instruction? ______

____

Does the benefit justify the cost? ______________________________________

FIGURE 4-2 *(continued)*

three aspects that should be considered in any program: instructional, media, and usage.

The instructional aspect of program selection refers to the instructional intent of a program. What type of software is it? Is it drill-and-practice, tutorial, or simulation? A drill-and-practice program may provide very good independent practice, but a child must possess the concepts and skills necessary to engage in the independent work. Tutorials may cover several steps in the teaching sequence. Simulations may be particularly good for motivation and for extending ideas and problem-solving skills. Naturally the content and its presentation must be accurate. You should remember that a computer can show only pictures or representations of objects, so children must have already had the concrete, manipulative experiences needed to make a program meaningful. By identifying what a program does, you can determine what you must do to prepare students for it and the proper follow-up activities to use following the CAI.

The technical aspect of selection refers to how well the program makes use of a computer's capabilities. The most important media characteristic of this aspect is the interactive quality of a program. Interactivity is the ability of the program to involve a student in the learning process and to react to what the learner does. The most sophisticated programs seem almost able to carry on a conversation with a child. The more natural and engaging the conversation or responses between learner and computer, the better is the interaction. A program with unclear directions or poor responses to students creates frustration by not giving needed information. Other media aspects to consider include how well the program makes use of color, sound, animation, and graphics, and its speed. Color monitors are widespread, so effective use of color is an important aspect of a computer program. Some programs use sound to motivate students and inform them of progress. The quality of the sound and an ability to turn it off, as well as to increase or decrease its volume, are important considerations for programs to be used in a classroom. Pictures and other graphics should be meaningful to children in the intended audience. The speed with which a program presents material or responds to a child should not be so fast that a child cannot keep up; yet it should not be so slow that a child becomes restless while awaiting a response.

You should also consider how a program will be used. Does the program fit instructional goals? Does the program come with good documentation and suggestions for its use? Good written information, or documentation, makes the program easier to use than when there is none. Does the program meet a unique need that is not met by other classroom materials? Is the cost of the program commensurate with its likely use? Can the program be used for small-group or whole-class instruction, or is it good for only one or two children at a time?

It will take 30 to 45 minutes with a program at a computer to complete the evaluation form. A recommendation regarding the program can then be made. Software review services provide reviews against which to compare your review; and teachers who lack the time or access to evaluate programs can rely on an evaluation service's review as a basis for a recommendation. *Software Reviews On File*[11], *Educational Software Report*[12], and *TESS*[13] are three services that provide reviews of software. Other reviews can be found in the *Arithmetic Teacher* and computer magazines. *Family Computing*[14] reviews home and educational software each month, giving information that you can use to advise parents about selecting programs for their children. Some teachers prepare newsletters to advise parents about how calculators and computers are used in the classroom. Recommendations about worthwhile programs can be made in such a newsletter.

LOGO

The programs already discussed offer little or no opportunities for children to be creative as they work with a computer. With most programs, children merely respond to questions and problems posed by the computer. Logo, a computer programming language, gives children a means to develop their own programs. Logo was developed over a fifteen-year period in the Artificial Intelligence Laboratory at the Massachusetts Institute of Technology by a group of mathematicians, computer scientists, and educators. Seymour Papert, a mathematician who studied with Jean Piaget, was one of Logo's developers and remains one of its primary advocates. He believes that Logo gives children a powerful tool for dealing with many sophisticated mathematical ideas using relatively few commands.[15]

By using the commands FORWARD, BACK, RIGHT, and LEFT, a child can move a triangular marker, or "turtle," around a monitor's screen. The turtle draws a line as it moves. Shapes and pictures can be drawn once a child learns to control distances and directions. A major difference between Logo and drill-and-practice, tutorial, and simulation programs is that Logo permits the child to be creative at the computer. It allows complete control of the learning environment by the learner.

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[11] *Software Reviews on File*, 460 Park Avenue, New York, NY 10016.

[12] *Educational Software Report*, Scholastic Magazines, PO Box 644, Lyndhurst, NJ 07071-9985.

[13] *TESS (The Educational Software Selection)*, compiled by the EPIE Institute, Box 839, Water Mill, NY 11976, and published by Teacher College Press, Columbia University, New York, NY.

[14] Scholastic Publications, 730 Broadway, New York, NY 10003.

[15] Seymour Papert, *Mindstorms: Children, Computers, and Powerful Ideas* (New York: Basic Books, 1980).

As children learn about Logo, they begin to challenge themselves with more involved projects. A child frequently learns to make a square as an early Logo activity. Success with the square leads to a desire to make triangles, pentagons, hexagons, and other plane figures. The length of sides and degrees in each angle must be determined. Trial-and-error processes lead many children to eventually figure out the "Rule of 360." The Rule of 360, also known as the "Total Turtle Trip Theorem," holds that the sum of degrees a turtle turns while forming angles for a closed figure is 360. Some older students will see this generalization quickly, but most children will need time and teacher guidance to understand it.

Logo has about 200 commands, but the language is not limited only to the built-in commands. A child can add new commands by defining procedures. For example, a square with each side 50 turtle steps long can be drawn by typing the commands FORWARD 50 RIGHT 90 four times. This square can also be made by using the REPEAT command, which saves much typing:

```
REPEAT 4[FORWARD 50 RT 90]
```

Another, even better way to form the square is to define a procedure for doing it:

```
TO SQUARE
REPEAT 4 [FD 50 RT 90]
END
```

Now the computer has been programmed to draw the square each time the new command SQUARE is given. Once children learn to make a square, they usually want to define procedures for drawing other figures.

Figures can be combined to form larger drawings. One problem-solving strategy discussed in Chapter 5 is to break a larger problem into smaller problems. Using Logo is an ideal way to practice this strategy. After children have learned to make individual plane figures, you can present a drawing of a figure that contains several figures and challenge children to determine how to use Logo to create a congruent figure.

Logo provides a means for exploring geometry and arithmetic. Besides the study of geometry as described above, Logo can be used to explore fractions, number lines, probability, and other topics in elementary mathematics. Activities related to these topics are presented in appropriate places later in this book. The *Logo Exchange*[16] is a journal featuring Logo activities, including many in mathematics.

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[16] International Council for Computers in Education, Department of Computer and Information Science, University of Oregon, Eugene, OR 97403.

COMPUTER AWARENESS

Computer-awareness activities are designed to acquaint students with computers and their uses and to provide information about career opportunities with computers.

A teaching unit about computers and career opportunities is frequently the means for presenting information to children in grades four through six. A comprehensive unit will include these topics: the history of computing and computers, hardware and software, computer applications, and career opportunities. Information should not be presented through lectures. Instead, children should become involved in library research, field trips, hands-on experiences with microcomputers, and other investigation-oriented activities.

History of Computing and Computers

- **a.** Make a collection of historical algorithms: the lattice method of multiplying, Russian peasant and finger multiplication, Babylonian multiplication and division, Egyptian algorithms for whole numbers and common fractions.[17]
- **b.** Make a collection of computing devices, including Napier's rods, Genaille's rods, abacuses (many versions have been used throughout the world); early mechanical digital calculators (some were developed in the 1600s); electric calculators; and contemporary calculators. Replicas, models, pictures, and actual devices can be made or obtained by children. Used mechanical calculators are sometimes available at little or no cost from business machine companies.
- **c.** Have children prepare reports on persons who developed computing devices, such as Blaise Pascal (1623–1662) who was in his teens when he developed a mechanical digital calculator for adding; Gottfried Leibniz (1646–1716), who invented a four-function calculator about 1671; and Charles Babbage (1792–1871), who designed the first modern calculating machine in the 1800s.[18] Ada Lovelace de-

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[17] See National Council of Teachers of Mathematics, *Historical Topics for the Mathematics Classroom* (Washington, D.C.: The Council, 1969), Chapter 3.

[18] See James R. Newman, *The World of Mathematics*, Vol. 1 (New York: Simon and Schuster, 1956), pp. 515–518, for a brief history of calculating machines. *Men of Mathematics* by E. T. Bell (New York: Simon and Schuster, 1937) contains biographies of Pascal and Leibniz that may be of interest to more able fifth- and sixth-grade children.

veloped ideas for a programming language for Babbage's machine. A programming language developed by the U.S. military services is called ADA in her honor.[19]

d. Collect pictures and information about computers. Include ENIAC, UNIVAC, and other early computers. Compare a microcomputer with the early machines. What are their similarities and differences?

2 Computer Hardware and Software

a. Collect information about input devices. Include pictures of keypunch, phone modem, keyboard, and other devices. Prepare a display of disks and tapes, keypunch cards, test-answer sheets, and other input paraphernalia.

b. Collect information about storage devices. How does a primary storage unit work? How is the capacity of a computer determined? Collect examples of secondary storage paraphernalia, such as keypunch cards, magnetic tape, disks, and Braille and other printouts.

c. How do central processing units work? Learn about the CPU's arithmetic–logic control and its memory units.

d. Investigate different programming languages. Why have different languages been developed? Which languages are most suitable for children? For teachers?

3 Computer Applications

a. Collect examples of how personal computers are used by individuals. Many children come from homes where there are computers. A parent, or the child, can demonstrate ways their computer is used. Game playing is the most common in-home use, but personal finances, recipe storage, address lists, and other applications are widespread. Do people produce their own programs, or do they use ready-made programs? What determines their choice?

b. Investigate ways computers are used in business, commerce, industry, sports, and government. How are computers used in banking and insurance? in supermarkets? in auto manufacturing? in college or professional football? in the post office? Visit places where computers are used to observe their operation and interview their operators.

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[19] David Moursund, *Introduction to Computers in Education for Elementary and Middle School Teachers*, p. 45.

c. Investigate and summarize student–school uses. Children who do not have access to computers can view a demonstration and experiment with CAI tutorials, simulations, word processing, and Logo or BASIC programming languages. Discuss how a computer helps administrators and teachers maintain school records.

d. Investigate the social impact of computers. Recognize that only people can program a computer. What potential problems exist in relation to the computer's impact on human rights and the way we live?

Career Opportunities

a. Make a list of computer-related occupations such as systems analysts, system designers, keypunch operators, programmers, and sales representatives. Emphasize that careers are open to all persons. Many video-game programmers are young people; some are still in high school. Invite a local programmer to tell about programming and to demonstrate games or other programs. Secure names of companies to contact from computer magazines and telephone book yellow pages.

b. Visit and hear speakers from places where computers are used. Children can generate a list of such places with input from parents and acquaintances.

c. Investigate requirements for entry into various computer-training programs. Contact local vocational schools, community colleges, and four-year and graduate schools to learn about training and educational opportunities. Determine the minimum requirements for each program.

KEEPING ABREAST OF COMPUTERS IN EDUCATION

New hardware and software suitable for school use are developed continually. The rapidity with which changes occur makes it impossible for this book to recommend particular brands of hardware or important hardware features. For example, when the fourth edition of this book was published in 1984, 64 kilobytes of memory was considered standard. Now 128 or even 256 kilobytes are needed to run many programs. The common keyboard was the standard input device; today a joystick, a mouse, or a touch pad or touch screen have joined the keyboard as common input devices.

Teachers who want to keep abreast of current hardware and software have a variety of information sources. They can talk to the educational specialists who work for local computer retail stores. Many universities have faculty members who specialize in instructional computing and are willing to share information. State education departments also have specialists in instructional computing. Professional conferences and workshops dealing with computers are common.

Many computer-using educators have organized groups to share information, assist newcomers, and join with others to present a unified voice in establishing goals and standards for computers in education. These support groups are excellent sources of information. An umbrella organization for many computer-user groups is the International Council for Computers in Education (ICCE)[20]. The ICCE publishes *The Computing Teacher*, a journal that presents a wide range of information about computers in education. An ICCE publication and other journals dealing with computers are included in the For Further Reading section of this chapter.

New software appears regularly. In a rapidly changing industry, some companies prosper and continue to operate, while others bloom and wither quickly. Appendix B contains the names of companies selected in expectation that they will continue to produce worthwhile mathematics software and will respond to inquiries about their products. You can write them directly to learn about their products and policies governing preview and distribution.

PREPROGRAMMED DEVICES

There are microprocessor-based devices other than calculators and computers that are used in the classroom. These are preprogrammed devices such as Rainbow,[21] on which a child presses keys to match answers in a drill-and-practice situation in mathematics (and other subjects). Cards present different operations; response time and scores are automatically recorded (Figure 4-3.) These devices do one job only and range in size from small hand-held devices to microcomputer-size units. Preprogramming limits the uses of these machines, which are not substitutes for calculators and computers. They do provide an alternate means of presenting drill-and-

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[20] International Council for Computers in Education, Department of Computer and Information Science, University of Oregon, Eugene, OR 97403.

[21] Available from Educational Insights, 150 W. Carob Street, Compton, CA 90220.

FIGURE 4-3

Preprogrammed learning aid. Photo courtesy of Educational Insights, Compton, California.

practice activities in mathematics that will appeal to some children. Suppliers of these devices include:

Texas Instruments, Inc.
Consumer Services
P.O. Box 53
Lubbock, TX 78408
Educational Insights
150 Carob Street
Compton, CA 90220

Summary

The National Council of Teachers of Mathematics has recommended that schools take full advantage of calculators and computers. Calculators can be used by children to solve problems that require complex or tiresome computation; to gain insight into numbers and operations; to investigate areas of mathematics they might otherwise avoid because of the amount of paper-and-pencil computation they would need to perform; and to learn to round numbers. Studies show that calculator use is not detrimental to children's computational skills and that calculators inspire them to do more problem-solving activities. Many children gain confidence in their problem-solving skills when they use calculators. A 4-function machine with an 8-digit readout is recommended for elementary-school children; memory, square-root, and percent keys are recommended for older children. Teachers need to develop a plan for using calculators before introducing them to children.

The first computer was placed in operation in 1945. Since then the cost and size of computers have been reduced and their power and reliability increased manyfold. Today it is feasible for schools at all levels to have one or more computers for administrative and instructional purposes. A microcomputer system consists of a computer, input and output devices, and a monitor. A printer may or may not be used. There are computer-assisted instruction (CAI) and computer-managed instruction (CMI) programs in many subject areas. Drill-and-practice, tutorial, and simulation are available in mathematics. These programs must be acted upon within prescribed limits. Programs can be obtained from commercial producers and computer consortiums. Teachers can write their own program with authoring systems, which are programming tools that guide the teacher through the writing process.

Logo is a language for computers developed at the Massachusetts Institute of Technology. It allows the student to interact with a computer in ways not possible with drill-and-practice, tutorial, and simulation programs. A child can create geometric figures and program a computer to recreate figures and designs generated by the child. It can also be used to further understandings and insights into other mathematics topics, including fractions and probability.

Computer awareness can be developed through a series of activities that deal with the history of computing devices and computers, hardware and software, computer applications, and career opportunities.

Preprogrammed devices provide an alternative method of giving children drill-and-practice activities.

Study Questions and Activities

1. Calculators are widely accepted and used by adults. Support for their use by elementary-school children is much less widespread. Ask several teachers and parents their opinions about calculators in the classroom and their own uses of calculators. List their concerns and contrast them to what has been found in research about calculator use (see For Further Reading section). What is your position on calculator use in the elementary school?
2. Grayson Wheatley (see For Further Reading section) believes much time-consuming computation should be completed with a calculator. He also suggests that time spent on topics such as long division is unnecessary and could be better spent on other topics in mathematics. Look into a fifth- or sixth-grade textbook to see how much emphasis is

placed on a long division algorithm and practice. Recall your own experience with such computation in fourth, fifth, or sixth grade. Do you agree that the calculator would be better for completion of these problems?

3. Investigate the policies and programs of a local school district with regard to computers and calculators. How are they currently being used in the mathematics program? Can calculators be used on homework and tests? What software is being used with computers? In your judgment, does the school district make full use of the computer and calculator as recommended by the National Council of Teachers of Mathematics?
4. Conduct a survey of a group of children to determine how many have calculators and computers at home. How do they use calculators? What do they do with computers at home? Ask them to name their favorite software for the computer. If computers are used at school, determine what the children like or dislike about the programs being used.
5. In many schools, children are expected to memorize much of the mathematics that they learn. Examine a scope-and-sequence chart for a mathematics textbook series and compile a list of the mathematics that students in such schools might be asked to memorize by the time they complete sixth grade. Include all areas of the mathematics curriculum. Examine the topics included in a computer drill-and-practice program and compare the topics there with the ones from the textbook. Do the computer program topics match the textbook topics? Do you agree with David Moursund that programs of this sort do not justify the expense of a computer system? Give arguments to support your answer.
6. Evaluate three pieces of software designed for elementary mathematics. Use the form from this chapter (Figure 4-2) or other review forms that might be available to you. Which of the programs do you enjoy the most? Are the programs primarily drill-and-practice, or do some of them build concepts or encourage problem solving and thinking?
7. Several computing magazines written for teachers or families are mentioned in this chapter. Scan several of these magazines. Write down all the examples of mathematics software you find in the magazines. Find a review of one elementary mathematics program to read. Based on the review, would you consider buying the program? What additional information would you want to know before buying the program? You may want to fill out inquiry cards in the magazine to obtain additional information about software from several companies.
8. Spend one or two hours working with Logo. If you are new to computer use, you may need some help to get started. Students who have been working with Logo are often ready to show you how to get

started. Have them show you what they have learned, and have them demonstrate some of their pictures and projects. What kinds of mathematical learning and problem solving do you find as you work or observe children?

For Further Reading

Arithmetic Teacher, XXXIV, No. 6 (February 1987). This theme issue on calculators features fifteen articles on curricular questions and instructional activities and updates the 1976 theme issue on calculators.

———, XXX, No. 6 (February 1983). This issue focuses on microcomputers in schools. The editorial argues for having children write programs in BASIC and Logo. Ten articles deal with general aspects of computers in schools, and three deal specifically with mathematics and computers.

———, XXIII, No. 7 (April 1976). This issue includes one editorial and eleven articles dealing with calculators. Information ranges from opinions, to how to use calculators, and classroom games and activities.

Bartalo, Donald B. "Calculators and Problem-Solving Instruction: They Were Made for Each Other," *Arithmetic Teacher*, XXX, No. 5 (January 1983), pp. 18–21. A ten-day sequence of activities involving children with calculators and problem solving is reported along with seven teaching suggestions. The lessons were designed to focus children's attention on problem-solving skills.

Bracey, Gerald W. "Computers in Education, What the Research Shows," *Electronic Learning*, II, No. 3 (November/December 1982), pp. 51–52, 54. Bracey reviews research about computers in education and discusses achievement, affective-motivational, and social outcomes. He also suggests some research that is needed.

Burt, Bruce H. *Calculators* (Reston, Va: National Council of Teachers of Mathematics, 1979). A compilation of over 50 articles from the *Arithmetic Teacher* and the *Mathematics Teacher* from 1976 to 1979 provides an essential resource for teachers using calculators.

Carpenter, Thomas P., and others. "Calculators in Testing Situations: Results and Implications from National Assessment," *Arithmetic Teacher*, XXVIII, No. 5 (January 1981), pp. 34–37. Some students who completed the mathematics test of the National Assessment of Educational Progress in 1977–78 were given calculators and instructed to use them as they did the test. The results showed that the calculators helped students at all age levels (9, 13, and 17) attain higher scores than peers who had no machines. The authors discuss recommendations concerning calculator use in classrooms.

Classroom Computer Learning. 2451 E. River Rd., Dayton, OH 45429. This monthly magazine contains timely articles on software, hardware, and educational computing topics.

Computing Teacher. 1787 Agate St., Eugene, OR 97403. This journal, published by the International Council for Computers in Education, was originally aimed at secondary teachers in computer science and mathematics. The content has broadened, however, to reflect wider use of computers in schools.

Duea, Joan, and Earl Ockenga. "Classroom Problem Solving with Calculators," *Arithmetic Teacher*, XXIX, No. 6 (February 1982), pp. 50–51. Examples of student-created problems and their uses in a classroom are presented. Calculators improve the quality and number of student-produced problems.

Electronic Learning. PO Box 644, Lyndhurst, NJ 07071-9985. This journal from Scholastic Publications provides up-to-date information about a variety of topics related to computers and schools, including reviews of equipment and software and discussion of instructional computing issues.

Family Computing. 730 Broadway, New York, NY 10003. Written for the home computer user, the wide range of information in this magazine is just as interesting and relevant for teachers and students. Reviews of new software are especially good. Available at newsstands.

Hembree, Ray. "Research Gives Calculators a Green Light," *Arithmetic Teacher*, XXXIV, No. 1 (September 1986), pp. 18–21. Calculators have an important place in today's mathematics class, according to research of the past ten years.

Journal of Computers in Science and Mathematics Teaching. PO Box 60730, Phoenix, AZ 85082. Although designed for secondary science and mathematics teachers, this journal also has articles of general interest on software, computer curriculum, reviews, and a Logo column.

Lappan, Glenda, Elizabeth Phillips, and M. J. Winter. "Powers and Patterns: Problem Solving with Calculators," *Arithmetic Teacher*, XXX, No. 2 (October 1982), pp. 42–44. A study of number patterns generated by working with the powers of numbers provides problem-solving experiences, practice with calculators, and application of calculator skills.

Logo Exchange. International Council for Computers in Education, Department of Computer and Information Science, University of Oregon, Eugene, OR 97403. Articles and activities with Logo for all grade levels and subjects are included in this journal.

Markuson, Carolyn, Joyce Tobias, and Tom Lough. "Logo Fever: The Computer Language Every School is Catching," *Arithmetic Teacher*, XXXI, No. 1 (September 1983), pp. 48–51. Describes advantages of Logo as a problem-solving language for elementary students.

Meyer, Phillis I. "When You Use a Calculator You Have to Think!" *Arithmetic Teacher*, XXVII, No. 5 (January 1980), pp. 18–21. The experiences of a group of sixth graders with calculators are described. The conclusion was that calculators helped the children think logically as they solved problems.

Morris, Janet P. "Problem Solving with Calculators," *Arithmetic Teacher*, XXV, No. 7 (April 1978), pp. 24–25. Describes several uses of the calculator. Included are exploring number patterns, discovering relationships, practicing mental estimation, reinforcing inverse operations, problem applications, developing problem-solving techniques, and individualized exploration and enrichment.

Moursund, David. *Calculators in the Classroom: With Applications for Elementary and Middle School Teachers* (New York: John Wiley & Sons, 1981). This semiprogrammed book presents useful information about calculators and how to use them. Examples of activities for elementary-school children are given.

———. *Introduction to Computers in Education for Elementary and Middle School Teachers* (Eugene, Oreg.: International Council for Computers in Education, 1981). The book's goals are to help teachers cope with computers in their schools, give ways to help children cope, and provide a foundation for learning more about computers.

———. *Teacher's Guide to Computers in the Elementary School* (Eugene, Oreg.: International Council for Computers in Education, 1980). This small book summarizes information about computers and their uses in elementary schools. It is good for a quick overview.

Reys, Robert E. "Calculators in the Elementary Classroom: How Can We Go Wrong?" *Arithmetic Teacher*, XXVIII, No. 3 (November 1980), pp. 38–40. This author favors the use of calculators. He discusses ten ways we can go wrong if they are used improperly.

School Science and Mathematics. 126 Life Science Building, Bowling Green State University, Bowling Green, OH 43403. Published by the School Science and Mathematics Association, this journal includes software reviews on science and mathematics as well as many general articles on activities and issues in mathematics education.

Teaching and Computers. PO Box 2040, Mahopac, NY 10541. This magazine by Scholastic Publications is aimed at elementary teachers interested in computers and includes articles about all elementary subjects.

Wheatley, Grayson, "Calculators in the Classroom: A Proposal for Curricular Change," *Arithmetic Teacher*, XXVIII, No. 4 (December 1980), pp. 37–39. Wheatley believes calculators should be used more than they are in mathematics classes. He recommends four areas of work that should receive reduced attention and six areas that need greater emphasis.

Wiebe, James H. "Using a Calculator to Develop Mathematical Understanding," *Arithmetic Teacher*, XXIX, No. 3 (November 1981), pp. 36–38. Understanding of counting, estimating, and computation can all be improved through calculator work. Examples of activities are given.

Winner, Alice-Ann, and Margo D. McClung. "Computer Game Playing—'Turn-On' to Mathematics," *Arithmetic Teacher*, XXIX, No. 2 (October 1981), pp. 38–39. The authors discuss the value of computer games, such as Blackjack, Guess My Number, and UFO, to fourth and fifth graders. They believe that concepts are learned and positive attitudes developed as games are played.

Chapter

5

Problem Solving

In Chapter 5, you will read about

1. the importance of problem solving as a goal in mathematics
2. twelve problem-solving strategies and guidelines for using them
3. problems that illustrate different problem-solving strategies
4. examples of realistic problems for children to solve in the classroom
5. problem simulation using the computer
6. suggestions for helping children understand written word problems
7. using children's own problems for problem solving
8. the role of estimation and mental computation in problem solving and how to develop and encourage estimation skills

The task of preparing students for roles in society was once much simpler than in schools today. In a slow-changing society, mathematical skills necessary for business, industry, and home were easily determined and taught. But over the years, occupational skills and mathematical requirements have expanded and become less definite. A fast-changing technological world offers few occupations for which mathematical requirements can be explicitly defined and developed during the school years. Today's students must understand a wide range of concepts and skills and must learn problem-solving strategies that have broad applications.

Problem solving is a primary goal of mathematics in schools. The National Council of Teachers of Mathematics put problem solving at the top of its agenda for mathematics in schools during the 1980s.[1] According to the California State Department of Education, "Instruction in mathematics must be planned to maximize each student's experience in solving challenging real-world and abstract problems."[2]

The term *problem solving* is used in this book to describe the actions an individual takes in any problem situation for which steps to an immediate solution are not apparent but for which a solution does exist. Before a problem can be said to exist, the individual must be committed to finding a solution. A student must understand that problem solving is a process, with solutions coming as the result of applying strategies in appropriate ways. At the same time, the learner must also recognize that the attitude "I can figure this out" is also an attribute of the successful problem solver.

When a broad definition of problem solving is used, word problems must be recognized for what they are: exercises in vicarious problem solving. They supplement, but cannot replace, the real problems children encounter during their in- and out-of-school activities. An alert teacher will see to it that children have many problem-solving experiences by planning art, science, social studies, and other lessons so that problems requiring mathematical solutions are included. Though out-of-school activities can also be used as a source of problem-solving experiences, a teacher has less control over them. Children's experiences in other subjects and extraschool activities are not likely to be sufficient to develop all the strategies children need. Textbook and other written problems can be used to supplement and provide practice with the strategies discussed in this chapter.

A problem for one person is not necessarily a problem for another. Schoolchildren face many situations that, on first encounter, are problems but that become familiar over time. For example, primary-grade children who gather data about their favorite cartoon characters need help in organizing and developing charts and graphs so they can share their information.

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[1] National Council of Teachers of Mathematics, *An Agenda for Action: Recommendations for School Mathematics in the 1980s* (Reston, Va.: The Council, 1980), p. 1.

[2] California State Department of Education, *Mathematics Framework for California Public Schools Kindergarten Through Grade Twelve* (Sacramento: The Department, 1985), p. 13.

Upper-grade children who have had many table-making and graph-making experiences will routinely prepare a table or graph with little or no adult assistance to present data they gather. One important reason for stressing problem solving is to help children master skills and strategies so they can attack unfamiliar situations in routine ways rather than face each one as a new problem to be solved.

Twelve strategies are presented in this chapter. Each strategy is accompanied by one or more problems. You should try to solve each problem before you read its solution, which appears at the end of the chapter. This chapter also discusses a four-step problem-solving process, suggestions for expanding problem-solving activities, and ways to help children learn to deal with textbook word problems and other written problems. Estimation skills are also discussed.

PROBLEM-SOLVING STRATEGIES

When faced with a problem, a person must explore the situation and select strategies that can be used to solve it. A good problem solver has a repertoire of strategies from which to choose. The following strategies apply to a wide variety of problems:

- Look for patterns.
- Draw a diagram or picture.
- Make a model.
- Construct a table or graph.
- Guess and check.
- Account for all possibilities.
- Act it out.
- Write a mathematical sentence.
- Break a problem into smaller parts.
- Restate the problem.
- Identify wanted-given information.
- Break set, or change your point of view.

Keep the following points in mind when you are presenting strategies to children.

- Most strategies apply to a variety of problems. No strategy has a "best" type of problem to which it always applies.

- A problem may be solved in several ways. It is not always necessary to use a particular strategy for a given problem.
- A strategy may be applied in different ways to different types of problems.
- Children will not reach the same level of sophistication in using each strategy.
- During the time children are learning strategies, processes of selecting and using strategies are as important as correctly solving a problem. Children should receive credit for choosing and using appropriate strategies, even if their solutions are not entirely correct.
- All children need opportunities to learn and use problem-solving strategies.

Look for Patterns

Patterns appear in many natural and person-made objects and situations. They are prominent in mathematics, so finding and using patterns has wide applicability in mathematical problem solving. Kindergarten and primary-grade children can find patterns in strings of colored beads, arrangements of blocks, and beats of a drum. Later, primary-grade children should study and find number patterns in sequences such as these: 0, 2, 4, 6, . . . , 15, 20, 25, 30, . . . , and 16, 15, 14, ... They can complete patterns begun by others and create their own. Young children investigate patterns in the designs and colors of T-shirts in the following activity:

PROBLEM 5-1

T-Shirt Patterns

- Select a child who is wearing a T-shirt that has a simple design, such as one with alternating blue-and-white stripes. Have the child describe the shirt's pattern.
- Select other children who have similar patterns on their T-shirts. Have each child describe his or her T-shirt's pattern.
- Tell each child to draw and color a picture of one of the T-shirts.
- Explain to the children how letters can be used to describe T-shirt patterns. The alternating blue-white pattern can be described as an a-b-a-b- . . . pattern. A pattern that alternates red-white-blue is an a-b-c-a-b-c- . . . pattern. (Narrow stripes of any color that separate wide stripes do not count as part of a pattern. Thus the pattern for a shirt with wide stripes separated by narrow black stripes is defined by the wide stripes only.)
- Write letter patterns such as a-b-c-d-a-b-c-d-. . . , a-b-a-c-a-b-a-c- . . . , and a-b-b-c-a-b-b-c-. . . on three-by-five-inch file cards, one

pattern to a card. Give a card to each child. Each child uses crayons to color a pattern for the letters on his or her card.

Older children should work with more abstract and complex patterns. The activities that follow present four different pattern investigations:

PROBLEM 5-2

Arithmetic Patterns

An arithmetic sequence is one in which the pattern has a constant between each number. The even numbers 0, 2, 4, . . . form an arithmetic sequence because 2 is added to each number to give the succeeding number. (Children should use calculators to do the computation required to solve these problems.)

- Write the sequence 1, 2, 3, 4, . . . on the chalkboard. Ask a child to tell what the hundredth number in the sequence is. (100)
- Write the first few odd numbers on the chalkboard: 1, 3, 5, 7, ... Have a child identify the sequence and name the next several numbers in it.
- Ask the children to name the hundredth number in the sequence. (199)
- Write the first few numbers of this sequence on the chalkboard: 4, 7, 10, 13, ... Have a child identify the constant in this series and name the next several numbers.
- Write each of these sequences on the board and repeat the activity with them:

  3, 6, 9, 12, . . .
  3, 9, 15, 21, . . .

- Ask the children to name the hundredth term in each of the final three sequences. (301, 300, 597)
- Help the children to derive a rule for determining any given number in an arithmetic sequence.

Number patterns similar to those in Problem 5-2 are presented in a computer program called *King's Rule*.[3] Students figure out increasingly difficult number patterns to progress through the castle to the Royal Suite. The game allows them to see a number of examples that follow a mystery rule, then to create and test their own patterns. Players who reach the Royal Suite will have demonstrated exceptional skill in solving unique pattern puzzles.

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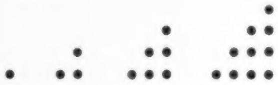
[3] Sunburst Communications Inc., 39 Washington Ave., Pleasantville, NY 10570.

The patterns for the next two problems are determined through investigations with manipulative materials. In the first, disks such as poker chips are used; the second uses a puzzle called "The Towers of Hanoi." Children in grades three through six can find the patterns in these two problems. Mature fifth- and sixth-grade children will be able to derive a rule for finding any given number in each pattern's sequence.

PROBLEM 5-3

Triangular Numbers

Triangular numbers were first studied by the Greeks, who were interested in relationships between numbers and geometry. The dot triangles illustrate the first 4 triangular numbers.



Write the first 4 triangular numbers.

Draw a picture of the next triangular number. Add this number to your list.

Study your list to see if you can determine a pattern in it. Write any pattern you find.

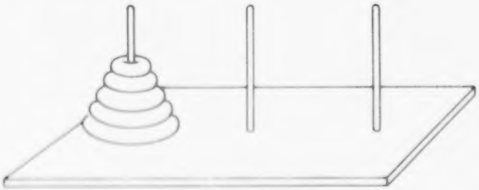
List the next 3 triangular numbers.

Determine, if you can, a general rule for finding the total number of dots in a triangle with any number of dots along one side. Write your rule as a formula. Use your rule to determine the number of dots in a triangle with 15 dots along one side.

PROBLEM 5-4

The Towers of Hanoi

The Towers of Hanoi is an ancient puzzle in which rings are moved from one post to another. In the picture, 5 rings are on one of the 3 posts. The problem is to determine the least number of moves required to move the 5 to either of the other posts.



Rules prescribe how moves are made. A larger ring can never be placed on top of a smaller one; and at the end, the rings must appear as they did at the beginning. (Use the third post to store in-transfer rings as the moves are made.) Problem: What is the minimum number of moves required to transfer the 5 rings?

How many moves are required to move 1 ring? Two rings? Three rings? Four rings?
Do you see a pattern you can use to determine the number of moves for 5? List any patterns you have found.
How many moves do you think are required to move 6 rings? Seven rings?
Determine, if you can, a general rule you can use to find the minimum number of moves required to transfer *n* rings. Use your rule to determine the number of moves required for a dozen rings.
Compare your rule and answers with those of a classmate.

The following problem is just for fun. It is not an arithmetic sequence, and there is no general rule for determining any given number in it.

PROBLEM 5-5

A Mystery Sequence

Identify the next 5 numbers in this sequence:

3, 4, 6, 8, 12, 14, 18, 20, 24, 30, . . .

Draw a Diagram or Picture

Diagrams range from a number line for the whole numbers 0 through 20 on which a first- or second-grade child figures out answers to simple addition and subtraction combinations, to more complex Venn diagrams such as the one in Problem 5-9, which was developed by fifth- and sixth-grade children. Children should be introduced to diagrams and pictures early and use them frequently throughout their school years. Diagrams and pictures are one means of making the transition from manipulative activities to abstract symbols. They will help you evaluate children's understanding of a problem situation; a child who is able to translate the situation into a clear diagram or picture demonstrates an accurate perception of the problem. A simple picture can be used for the following problem, which is suitable for third- or fourth-grade children. The second problem can be diagramed with lines. Venn diagrams are useful for the third and fourth problems.

PROBLEM 5-6

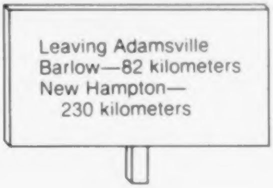
How Many Birds?

What is the smallest number of birds that can fly in a straight-line formation so that 2 birds are in front of a bird, 2 birds are behind a bird, and 1 bird is between 2 birds?

PROBLEM 5-7

Leaving Adamsville

This road sign appears at the city limits of the town of Adamsville:



How many miles must you travel until you are halfway between Barlow and New Hampton?[4]

A Venn diagram is a special diagram that is used to show relationships between and among members of 2 or more sets. Children should learn about Venn diagrams as they work with sets of attribute blocks and other manipulative materials. Activities 6-15 and 6-16 show how Venn diagrams can be used as children do classification activities. Primary-grade children

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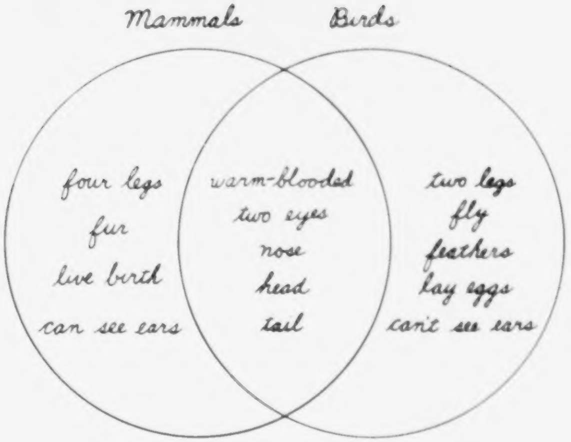
[4] Adapted from Carole Greenes and others, *Techniques of Problem Solving: Problem Card Decks* (Palo Alto: Dale Seymour Publications, 1980), Card 48-C.

FIGURE 5-1

First grader's Venn diagram listing attributes of mammals and birds



can use Venn diagrams to show such things as the common physical attributes of animals they study. Figure 5-1 shows a Venn diagram that first graders made to show some of the attributes of birds and mammals; the attributes common to both animals appear in the intersection. Use Venn diagrams to solve each of the next two problems.

PROBLEM 5-8

The Dog Kennel

John and Joe owned 4 dogs jointly. John owned 3 himself. Altogether they had 12 dogs in a kennel. How many did each have?[5]

PROBLEM 5-9

The Math Teacher

A math teacher conducted a survey in her classes. Forty-two students chewed gum in class; 55 talked incessantly; and 25 fell asleep during class. Four people both talked and fell asleep but didn't chew gum. Six students fell asleep and chewed gum but at least didn't talk. Forty students only talked. Ten students talked and chewed gum but didn't fall asleep. How many students were there in all? How many students talked and chewed gum and fell asleep (UGH)?[6]

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[5] Adapted from Maria Marolda, *Attribute Games and Activities* (Palo Alto: Creative Publications, 1976), p. 141. (Used by permission.)

[6] Maria Marolda, p. 141. (Used by permission.)

3 Make a Model

A model is similar to a drawing except that objects rather than drawings are used. Play money is frequently used in primary grades to give children practice with buying and selling, and model clocks are used in time-telling activities. Three-dimensional models of cubes, pyramids, and other geometric solids are used as children learn about these figures. An advantage of models over diagrams is that they can be moved about as the learner deals with a problem. Plastic robots used with a centimeter ruler are good models for the problem that follows, while disks and squares of papers can be used for the second and third problems, respectively.

PROBLEM 5-10

A Line of Robots

Five robots are walking down the street. Bocko is 25 meters ahead of Rocko. Locko is 10 meters behind Bocko. Mocko is 5 meters behind Locko. Mocko is 15 meters behind Socko. List the robots in order from left to right.

PROBLEM 5-11

Cycle Riders

I counted 7 cycle riders and 19 cycle wheels go past my house Saturday morning. How many bicycles and how many tricycles passed my house?[7]

PROBLEM 5-12

Stamp Collectors

Marcia has three times as many stamps as her brother. If her brother had 8 more stamps, they would have the same number of stamps. How many stamps does Marcia have?[8]

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[7] Ohio State Department of Education, *Problem-Solving—A Basic Mathematics Goal, Book 1, Becoming a Better Problem Solver* (Columbus: The Department), p. 16. (Used by permission.)

[8] Carole Greenes, Card 44-C.

Construct Tables or Graphs

Statistics are used by planners in many fields. An architect uses information about a new building's occupants to plan rooms, furnishings, and safety features; an insurance actuary uses data about births, deaths, and ages to establish insurance rates; and a shipping agent uses data about production schedules and product destinations to plan shipping schedules.

The data children use is less complex than that in the examples above. Nevertheless opportunities abound for children to deal with realistic problems that require the use of tables and graphs. Chapter 14 discusses tables and graphs in greater detail. Problem 5-13 illustrates a graph-making activity for primary-grade children.

PROBLEM 5-13

Our Favorite Ice Cream Flavors

"What ice cream does each of you like best? When each of you tells me the flavor you like best, I will give you a paper square that has the same color as your favorite ice cream."

- Give brown squares for chocolate, cream-colored for vanilla, and so on.
- After the squares are distributed, have the children paste them in columns by color on a large piece of butcher paper on which a baseline has been drawn. Make certain that gaps are not left between squares in each column. Write the name of the ice cream represented by each color beneath the proper column.
- Ask questions, such as: "What ice cream is most popular?" "Are there any flavors that are equally popular?" "How many fewer children like vanilla ice cream than like chocolate ice cream?"

While several strategies can be used to solve Problem 5-14, you are to use a graph to solve it.

PROBLEM 5-14

Christmas Ornaments

Our class is going to make Christmas ornaments to sell. Each ornament will have 7 gold stars. How many stars will be needed for 16 ornaments?

A series of tables that list different possibilities can be used to solve the following problem:

PROBLEM 5-15

Two Unique Rectangles

There are only 2 rectangles whose sides have measures that are whole numbers and whose area and perimeter are the same number. What are they?[9]

Guess and Check

One of the least appreciated strategies is *Guess and check*. Guessing for the solution of a problem is frequently associated with aimless casting about for an answer. We do not advocate blind guessing, but we wholeheartedly do encourage educated guesses.

What is the difference?

An educated guess is based on the problem solver's careful attention to pertinent aspects of the problem, though not necessarily in a formal manner. This information is assimilated with his previous experiences to produce a hypothesized answer that the guesser has some reason to expect to be "in the right ball park."

As a strategy, the key element is *and check*. In other words, the problem solver makes an educated guess *and then checks* it against the problem conditions to determine how to improve his guess. He repeats this process until he has an acceptable answer.[10]

This problem is a good one for children to solve with the guess-and-check strategy.

PROBLEM 5-16

Buying Apples

Large apples cost fifty cents each. Smaller apples cost forty cents. Jamie buys some apples and has twenty cents change from three dollars. How many large apples and smaller apples did Jamie buy?

Account for All Possibilities

This strategy works well when there is a limited number of possibilities to be considered. Charts or tables are frequently used to organize information when this strategy is used. In the first problem, use a chart to list all of the possible combinations of ice cream and toppings. Use the chart to show a Cartesian or cross product for the situation. (See pages 352–355 in Chapter 10 for further discussion of Cartesian products.)

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[9] Ohio State Department of Education, p. 24. (Used by permission.)

[10] Ohio State Department of Education, p. 17. Emphasis in original. (Used by permission.)

PROBLEM 5-17

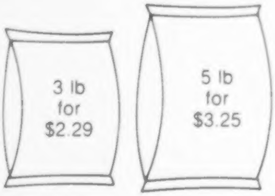
The Ice Cream Parlor

An ice cream parlor has 4 flavors of ice cream and 6 flavors of topping. Are there enough combinations of ice cream and topping so that 19 children can each have a different sundae?

To solve the following problem, make a table to show all of the possible ways you can combine bags of fertilizer to get 17 pounds.

PROBLEM 5-18

Buying Fertilizer



You need 17 pounds of fertilizer, which comes in three-pound or five-pound bags. How many bags of each size do you buy to obtain at least 17 pounds for the lowest cost?[11]

Act It Out

The acting-out strategy is different from dramatization. Children who dramatize duplicate a real-world situation without solving a problem. In the primary grades, children dramatize a buying situation when they are in a classroom store stocked with empty food and household product containers that are used to give children experiences in buying and selling the store's products. Older children dramatize a banking situation when they earn credits for accurate and on-time work, good behavior, courtesy, and so on, and "deposit" these credits in a classroom "bank." Credits are "withdrawn" to buy free time, school supplies, or other rewarding objects. Both situations give children opportunities to learn worthwhile consumer skills, but they solve no problems.

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[11] Ohio State Department of Education, p. 11. (Used by permission.)

The acting-out strategy is used as a means of solving problems and is often spontaneous and is usually not very elaborate. Consider the two situations that follow. What props or characters are needed for each one?

PROBLEM 5-19

Handshakes

You are walking down the street when you and 5 friends meet at a street corner. If you and each friend shake hands with every person in the group, how many handshakes are made?

PROBLEM 5-20

Horse Trading

A woman buys a horse for sixty dollars, sells it for seventy dollars, buys it back for eighty dollars, and then sells it for ninety dollars. How much money does the woman make or lose in the horse-trading business?[12]

Write a Mathematical Sentence

Children at all levels should be encouraged to write mathematical sentences appropriate for problem situations. A sentence serves two purposes: It shows you that a student understands a problem situation; and it shows the student the computation that is needed to solve a problem. A sentence can also be used to verify the solution obtained with another strategy. For example, in the "Buying Apples" situation (Problem 5-16), a child can write these sentences to verify the "guess-and-check" strategy:

$$50 + 50 + 50 + 50 + 40 + 40 = 280$$
$$300 - 280 = 20$$

The following problem situations illustrate types that are suitable for solution with mathematical sentences.

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[12] Ohio State Department of Education, p. 15. (Used by permission.)

PROBLEM 5-21

Airplane Trip

An airplane flew from San Francisco to Los Angeles (347 miles), from Los Angeles to El Paso (701 miles), from El Paso to Houston (676 miles), and from Houston to New Orleans (318 miles). How far did the airplane fly?

PROBLEM 5-22

Buying Groceries

Jack bought these items at the grocery store: One loaf of bread for $1.26, 2 pounds of apples at fifty-nine cents a pound, and 3 cans of cat food for thirty-three cents a can. How much money did the purchases cost?

Break a Problem into Smaller Parts

A complex problem can be broken into smaller parts to make it easier to solve. An example of where this strategy might be used is Problem 5-22, which can be broken into 3 separate problems. Once the answer to each problem has been determined, the 3 numbers can be added to give the complete answer. The complex figure in Figure 8-10 will need to be broken into separate figures in order to reconstruct it. Use this strategy to solve the following problem.

PROBLEM 5-23

Measuring Shadows

Indirect measurements of objects can be done by measuring an object's shadow and comparing the measure of the shadow with another object's measure. (This is a good activity to complete with a calculator.)

Work with a partner. Determine your height and your partner's height to the nearer centimeter.

Go outside and measure the length of your shadow and your partner's shadow to the nearer centimeter when you stand at full height in the sun. (Stand on a sidewalk or other hard surface so your measurements will be as accurate as possible.)

Find an object that is taller than you and your partner, such as a tree or flagpole. Measure the length of the shadow of that object to the nearer centimeter.

Return to the classroom. Write the numbers that make a proportion of your height and your shadow's height. Use that proportion and the known length of the object's shadow to determine the object's height. State the height of the object in meters. Is your answer reasonable?[13]

Restate the Problem

Sometimes a problem is not understood at the time it is first read. A careful second reading followed by a restatement of the problem often makes it clear. Children who can restate even the simplest problems in their own words demonstrate understanding. There are times when restatement leads directly to a solution. Read each of these problems carefully, then determine their solutions.

PROBLEM 5-24

At the Five and Dime

I bought some items at the Five and Dime store. All the items were the same price, and I bought as many items as the number of total cents in the cost of each item. My bill was $2.25. How many items did I buy?[14]

PROBLEM 5-25

Wine and Cork

A bottle of corked wine costs ten dollars. The wine is valued at nine dollars more than the cork and bottle. What is the value of the wine?

Identify Wanted-Given Information

The wanted-given strategy is used when a person examines the problem to find out what is wanted and what is given. What is wanted may be a sum, a difference (unknown addend), a product, a quotient (unknown factor), or some other answer. What is given is often a pair (or more) of addends, a sum and a known addend, a pair of factors, a product and a known factor, or a combination of these. Activities for this strategy are discussed later in this chapter.

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[13] Adapted from Cathy Hardman, "Sun-Made Shadows," *The CMC ComMuniCator*, XI, No. 1 (California Mathematics Council), p. 12.

[14] Ohio State Department of Education, p. 29. (Used by permission.)

Breaking Set, or Change Your Point of View

Various factors can make a problem difficult. Most we have no control over, but one that is within our discretionary power is the perspective we take of the problem. Upon meeting most problems, we tend to quickly adopt a particular point of view. We speedily formulate a plan of attack and then implement it to determine whether it produces a solution. If our plan is not successful, the tendency is to return to the same point of view and adopt a new plan. This may be successful. But it may also involve a circle of faulty logic based upon an incorrect or nonproductive initial perspective.

The real culprit in this process is failure to examine adequately the problem initially. The better we understand the full nature of a problem, the more likely we are to choose a successful line of attack. However, it is human nature to progress rapidly through the problem-identification stage. We should therefore acknowledge this human frailty and provide a means to escape from a failure cycle created by an unsuccessful initial view of a problem.

We recommend the strategy CHANGE YOUR POINT OF VIEW. That is, cease current problem efforts, discard any previous notions about the problem, and try to redefine the problem in a completely different way. Unlike RESTATING THE PROBLEM, changing your point of view requires a major shifting of focus upon the problem components. Elements which were emphasized are now downplayed or perceived in a different manner. Other problem features are highlighted that previously may have been given very little consideration. We try to imagine how the problem state could be achieved in ways other than initially conceived.[15]

Apply the breaking-set strategy to each of the following problems.

PROBLEM 5-26

The Farmer's Trees

A farmer planted 10 trees in 5 rows of 4 trees each. How did she do this?

PROBLEM 5-27

Checkers

How could a man and a woman play 5 games of checkers and each win the same number of games without any ties?[16]

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[15] Ohio State Department of Education, p. 16. (Used by permission.)

[16] This and the tree problem are adapted from Robert J. Sternberg and Janet E. Davidson, "The Mind of the Puzzler," *Psychology Today*, XVI, No. 6 (June 1982), p. 37.

PROBLEM 5-28

Triangles

How can you arrange 6 new lead pencils to form 4 congruent equilateral triangles?

A FOUR-STEP PROBLEM-SOLVING PROCESS

A repertoire of problem-solving strategies helps a person be an effective problem solver. However, strategies alone are not sufficient. One must also possess knowledge of the problem-solving process itself. George Polya was a mathematician who studied problem solving during his career at Stanford University. His book *How to Solve It*[17] was first published in 1945 as a guide for university students. His four-step process is also useful for precollege students; a simplified version is advocated for elementary school children. Polya's four steps are

Understanding the problem
Devising a plan
Carrying out the plan
Looking back

Understanding the Problem

The first step is to understand the problem. Study the situation to determine what is given, what is to be found, and what is needed. Is the information that is given sufficient? If not, what more is needed? Perhaps a model or diagram will help. Do not forget about breaking set, or taking a different point of view.

PROBLEM 5-29

Sixty-three Dollars

How is it possible to have sixty-three dollars in bills without having any one-dollar bills?

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[17] George Polya, *How to Solve It* (Garden City, N.Y.: Doubleday, 1957).

PROBLEM 5-30

Candlestick Park

Could all of the students in our school be seated in Candlestick Park in San Francisco? Could all of the people living in Elk Grove, California, be seated in Candlestick Park? All of the people living in Sacramento, California?

- Analyze each of the preceding problems; then write statements that show you understand each one.

2 Devising a Plan

Once a problem is understood, a plan for solving it must be devised. Consider previously solved problems to determine if strategies used earlier can be used now. Is there a general rule that applies? Which strategies are appropriate for the problem? Simplify or restate the problem if necessary. Is the plan complete? Does it take into account all aspects of the problem? Will it yield a complete answer?

- Write a description of each of your plans for solving the preceding problems.

3 Carrying Out the Plan

As a plan is carried out, each step should be checked. Is all computation reasonable? Do charts and graphs contain all pertinent information? Are the data leading to useful answers? Are guesses reasonable ones? (Do not waste time on wild guesses.)

- If possible, complete the preceding problems.

4 Looking Back

According to Polya, this is a very important step. Teachers and children both need to go beyond the solution. Now is the time to discuss whether the result could have been derived differently. Was the chosen strategy the best or most efficient? Was it applied correctly? Also consider other past and potential problems to help children learn that there are classes of problems for which certain strategies and general rules are applicable.

- Think about the approach and process you used with each of the two problems. Tell whether you could have done anything differently to make your work more efficient.

CLASSROOM PROBLEMS

An alert teacher will be aware of the many opportunities that arise in the course of a school year to give children realistic problem-solving experiences. The following are examples of common situations that can be used by a teacher to give children opportunities to apply Polya's four-step process and problem-solving strategies.

- A class needs sashes for a PTA program. Each sash has 3 stripes of different colored material, and each stripe is four inches wide and forty-eight inches long. How many sashes are needed? How many yards of each color will be needed?
- Two computers are in the classroom. What is the best way to schedule children's use of the computers so that everybody has a turn with the least interruption of other activities?
- A class will make pizza for Parent's Night. How much will it cost to make 1 pizza? How many slices can be cut from 1 pizza? What is the best price to charge for 1 slice in order to make a fair profit? How many slices must be sold in order to make enough money to buy a copy of the *Print Shop* computer program?
- A science project dealing with the growth of plants under varying light conditions is planned. What measurements will need to be made? How often should measurements be made? What types of records should be maintained? How should the results of the project be reported?
- A system for rewarding good classroom behavior is to be established. Children will earn credits for good behavior, which will be "banked" for future redemption. What acts of good behavior will earn credits? How will the value of an act be determined? Who will award credits, and how will they be given? How will credits be recorded? Who will handle the recording and redemption of credits? For what "rewards" will credits be exchanged? How will the value of a "reward" be determined?
- A new surface has been put on the school's playground hard-top; all of the markings for game layouts have been covered. What game spaces should be laid out on the new surface? What is the best way to lay out the spaces to economize on space and provide for the least

possible interference among players of different games? How much paint will be needed? How much will the paint cost?

A class is investigating climatic changes in its region of the country. What is the average monthly rainfall for the past five years? What is the average yearly rainfall for the past five years? How do these averages compare with the same averages for the past 100 years? Does it appear that the average for any given month has changed? Does the average for a year appear to have changed? Make the same comparisons for temperature. What, if any, climatic changes appear to have taken place over the past 100 years?

A class is studying nutrition as part of a health unit. What number of calories is recommended for girls the age of those in the class? For boys? What distribution of the calories by food groups is recommended? What is the average daily intake of calories for each child in the class? How are the calories for each child distributed among the food groups? By how much does each child's calory intake vary from what is recommended?

SIMULATION PROBLEMS

Several computer programs offer opportunities for simulated problem solving. The two described here are examples. The *Whatsit Corporation*[18] is a simulated business. A group of 3 or 4 students are engaged in setting up and running a small corporation. They must make all the decisions about the product and its selling price and distribution. The documentation for this program gives many suggestions for extending the problems encountered in the simulation. The *Math Shop*[19] offers jobs in 10 different shopping-mall stores. While serving customers, students make decisions and use mathematics skills such as weighing gold or making change.

WORD PROBLEMS

You are encouraged to present many in-school activities that expose children to realistic problems. Teachers who do this know that many problem-solving strategies can be learned this way. There are times, however, when word problems must be used to broaden children's exposure to strategies.

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[18] Sunburst Communications Inc., 39 Washington Ave., Pleasantville, NY 10570

[19] Scholastic Inc., PO Box 7502, Jefferson City, MO 65102

Word problems also provide realistic settings in which to introduce new topics.

Results of the second *National Assessment of Educational Progress* mathematics test indicate that nine- and thirteen-year-old children could solve word problems involving a single operation nearly as well as they could compute answers to algorithms for the same operation. They were more skillful in computing answers and solving problems involving addition and subtraction with whole numbers than they were with multiplication and division.[20] The downfall of most students was work with less-familiar problems, such as multistep problems, and those with too much or too little information. This led Carpenter and his associates to recommend that more attention be given to such problems.[21]

In addition to the strategies and four-step process already discussed, attention should be directed to activities that deal with

Reading in mathematics
Children's own problems
Extraneous and too-little information

Reading in Mathematics

Children's achievement in mathematics is affected by their language development. Generally children who lack adequate language mastery have more difficulty in achieving success in mathematics than those whose mastery is good. Even children whose language development is normal may experience problems with reading in mathematics.

One aspect of the problem is the vocabulary of mathematics. Many words used in mathematics are already familiar in other contexts, so their mathematical meanings need special attention. For example, *base, difference,* and *product* have meanings apart from those used in mathematics. Other words, such as *array, quotient,* and *commutative,* may be new when children first encounter them in mathematics. Earp and Tanner suggest that both types of words should be developed through an oral-language process. They offer these suggestions:

> Much oral language should be used in mathematics classrooms. More time should be spent discussing mathematical ideas than is commonly used.
>
> Mathematics vocabulary should be taught in ways similar to those

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[20] Thomas P. Carpenter, and others, "Solving Verbal Problems: Results and Implications from National Assessment," *Arithmetic Teacher*, XXVIII, No. 1 (September 1980), pp. 9–10.
[21] Carpenter, pp. 10–11.

for teaching vocabulary in a reading lesson. Children should hear, see, and say mathematical words, pronounce them by syllables, and use them in written and oral sentences.

Mathematical statements in textbooks should be rewritten or paraphrased by both children and teachers.

The cloze procedure should be used. With this procedure, a blank replaces every *n*th word in a sentence or statement. Children complete each one by filling in the blanks with suitable words.[22]

The symbols of mathematics, that is, signs of operation (+, −, ×, ÷), of relation (=, <, >, ≠), and others, must be clearly understood. Abbreviations, such as cm, L, lb, and #, and commonly used Greek and Latin prefixes and suffixes, such as bi-, centi-, geo-, para-, -hedron, and -gon, should receive special attention. (Appendix D gives the meanings of commonly used Greek and Latin prefixes and suffixes.) A classroom chart to which each new symbol, abbreviation, prefix, and suffix is added when introduced will remind children of the meanings.

Children must learn to read mathematics slowly and carefully. Greater attention must be given to detail when reading in mathematics than when reading a story. (The first two steps of the four-step problem-solving process deal with this.)

Looking for clues in word problems is often taught as a strategy for determining which procedures to use. Examples of clue words within sentences are: "How many are there in all (or altogether)?" (Addition). "What is the difference?" and "How many are left?" (Subtraction). However, such clues are so limited in value as to be almost useless; many children use them improperly, thinking they suggest one operation when another is appropriate. The following problem illustrates how a clue word can lead to an incorrect answer.

PROBLEM 5-31 **Bob's Marbles**

Bob had 36 marbles. When his father gave him a new bag of marbles, Bob counted all of them and found he had 51 altogether. How many marbles did his father give him?

The word *altogether* in this problem incorrectly suggests addition. Another weakness of the clue-word strategy is that many problems do not include useful clues. If the clue-word approach is the only or primary strategy a child has, the child will often give up or resort to guessing.

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[22] N. Wesley Earp and Fred W. Tanner, "Mathematics and Language," *Arithmetic Teacher*, XXVIII, No. 4 (December 1980), p. 34.

Children's Own Problems

Personalized word problems have the advantage of matching children's interests and using settings and vocabulary with which they are familiar. Word problems are also valuable for checking children's understanding of mathematics.

The language–experience approach can be used to introduce primary-grade children to word problems. With this approach, you set up situations that children act out.

ACTIVITY 5-1

The Books

Begin with a stack of 13 books. Give 8 of them to a child, and keep 5. Discuss your actions. How many books did I have at first? How many were given to Bobbie? How many did I keep?

Have a child tell a story about the books; write it on the chalkboard or a chart.

Repeat activities of this sort over time until children are fluent in interpreting actions and telling stories about them.

Another activity is to have children make up problems for sentences that you specify.

ACTIVITY 5-2

Making up Word Problems

Write these sentences on the chalkboard: $12 - 8 = 4$ and $3 + 7 = 10$. Instruct the children to make up a word problem for the subtraction sentence and one for the addition sentence.

Have volunteers tell their stories. Discuss each one to see how it fits its sentence.

Children's stories may be written or oral, depending upon their writing skills.

The language–experience approach can be modified for older children. Instead of acting out a problem (the sizes of numbers used make this cumbersome), present a number sentence like 436 + 389 = 825. Discuss situations that could give rise to the sentence. At first, children's contributions may be limited, but with experience they will generate many situations. Practice should follow discussion. At first, present them with clearly specified conditions, such as in this example.

ACTIVITY 5-3

Subtraction Word Problem

Make up a word problem involving subtraction in which a two-digit number is subtracted from a two-digit number and for which regrouping is necessary. Example: 53 − 19 = 34.

As children gain experience in writing word problems, your directions can become less specific. Now you might direct them to make up a word problem dealing with an extraschool experience such as measuring a garden plot, doubling ingredients for a recipe, or ordering Girl Scout cookies. Encourage them to discuss this assignment with and solicit suggestions for problems from adults at home.

Children's problems should be shared. One way to share is to exchange problems during a work period. Discuss solutions before the period's close. Another way is to have children put each problem on a four-by-six-inch card for a classroom file. Children can work with these cards during problem-solving practice periods, as part of a planned mathematics contract, or as self-selected free-time activities. Over time, you will accumulate a collection of personalized problems to supplement or replace textbook problems.

Extraneous or Too-Little Information

Word problems with too much or too little information should be assigned and discussed from time to time. Each problem should be studied carefully to determine what information is given. Statements that do not contain questions or numbers provide practice in determining what is lacking. Here are examples of problems for these types of exercises:

> A car can go 17 miles on 1 gallon of gas. How far can it travel on 1 tank of gas?
>
> Jack had $5.00 in his bank. He then worked for 6 hours and earned $3.25 an hour. How much money did he earn for his work?

- An airplane has a top speed of 1100 miles an hour. It flew at this speed for 3 hours.
- Jack bought a new baseball glove. How much change did he receive?

Read each problem. Does it provide too much or too little information? Change each one in a way that you think makes it a good story problem.

ESTIMATION

Estimation skills have wide application in mathematics. Children need to learn how to make reasonable estimates when giving answers to problems and when computing, measuring, and determining quantities of objects.

Estimating Answers

Skill in estimating answers has gained importance as people have become aware of its use in problem solving. One part of the carry-out-the-plan phase of the problem-solving process is to check the accuracy of each step. Estimation is often used to do this. Whether a calculator or paper and pencil are used, estimation is useful for judging the reasonableness of one's work.

$$\begin{array}{r} 290 \\ 320 \\ 380 \\ +430 \\ \hline \end{array}$$

Before children can estimate answers, they must know how to round off numbers to the nearest ten, hundred, or thousand. To estimate the answer for the addition shown in the margin, a person must first round each addend to the nearest hundred. A number line such as the one in Figure 5-2 is useful for helping children see that 290 is nearer to 300 than to 200. When 290 is located on the line and the steps are counted between 290 and 200 and between 290 and 300, the reason for rounding off 290 as 300 becomes clear. After all four numbers have been rounded off, the answer can be estimated by adding only the hundreds.

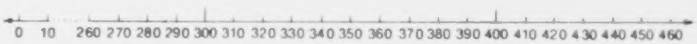


FIGURE 5-2
Number line used to help children round off numbers to the nearest hundred

When children in elementary school are taught to round off numbers to the nearest ten, they learn that the numbers 31, 32, 33, and 34 are rounded to 30; and that the numbers 36, 37, 38, and 39 are rounded to 40. Children can use a number line to see that 31, 32, 33, and 34 are closer to 30 than to 40; and that 36, 37, 38, and 39 are closer to 40 than to 30. When 35 is considered, however, the number line shows that it is as close to 30 as to 40.

Children will need to be given a rule for rounding off numbers that end in 5. A commonly used rule is to round off the numbers to the next higher ten. By applying this rule to the number 35, the children learn to round it off to 40.

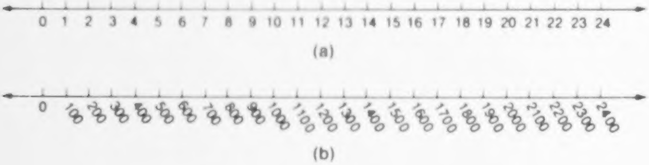


FIGURE 5-3

Number lines used to help children learn to round off numbers to (a) the nearest ten and (b) the nearest thousand

A number line similar to the one in Figure 5-3(a) is useful for rounding numbers to ten; and the one in Figure 5-3(b) can be used to round numbers to the nearest thousand. These lines can be made on calculator tape and fastened to the classroom walls. (A way to use a calculator for learning to round off numbers is explained in Chapter 4.) After the children have learned to round off numbers to the nearest ten, hundred, and thousand, help them work with numbers that are to be rounded off first to the nearest ten, then to the nearest hundred, and then to the nearest thousand. Activities like the following are useful:

ACTIVITY 5-4

Rounding Off

Round off each of these numbers to the nearest ten:

3291 4648 13,689

Round off each number to the nearest hundred.
Round off each number to the nearest thousand.

Children should use estimates to judge the reasonableness of computation. Encourage them to make estimates before answers are computed. One way to do this is to have them complete a practice exercise in stages. During stage one, the children estimate the answers for a set of problems, such as those in Problem 5-32. Discuss the children's estimates, then have them complete the computation as stage two of the activity. Have the children compare their estimates with the computed answers to see if any computed answers are unreasonable.

PROBLEM 5-32

Estimation

Estimate an answer for each of these examples:

$$\begin{array}{r}23\\+49\\\hline\end{array}\quad\begin{array}{r}18\\+21\\\hline\end{array}\quad\begin{array}{r}67\\+42\\\hline\end{array}\quad\begin{array}{r}63\\-21\\\hline\end{array}\quad\begin{array}{r}48\\-19\\\hline\end{array}\quad\begin{array}{r}61\\-18\\\hline\end{array}$$

Estimate an answer for each of these examples:

$$\begin{array}{r}16\\\times\ 3\\\hline\end{array}\quad\begin{array}{r}73\\\times 21\\\hline\end{array}\quad\begin{array}{r}978\\\times\ 48\\\hline\end{array}\quad\begin{array}{l}438 \div 11 = \square\\698 \div 35 = \square\end{array}$$

Give an estimated answer for each of these word problems:

- What is the cost of a 39-cent can of fruit and a 48-cent quart of milk?
- Each bus carries 28 passengers. How many passengers can 6 buses carry?
- There are how many 29s in 612?

Children may compute the problems or use calculators to determine exact answers against which to judge their estimates.

Sometimes an estimated rather than an exact answer will solve a problem. This is true of the following situations:

PROBLEM 5-33

More Estimation

Use estimation to solve these problems:

- Sally wants to buy a baseball glove that costs $12.95 and a baseball that costs $3.25. She has $18.00. Is this enough money to purchase the ball and glove?
- A jet plane can carry enough fuel to remain airborne for 5 hours of flight at top speed. If it flies at a top speed of 890 miles per hour, can it make a round trip of 5000 miles without having to refuel?

Estimation is used to help children learn to place the decimal point properly when multiplying and dividing with decimal fractions. Activities dealing with these applications are given in Chapter 12.

Estimating Measurements

Skills in estimating with measures have wide application. Instruction in this aspect of estimation must be planned as an integral part of children's measurement experiences. Estimation activities are presented in Chapter 13, which deals with measurement.

Estimating Quantities

A popular means of attracting customers to a business is to place a jar of gum balls (or similar objects) in a store's window along with an invitation to make a guess about the number of gum balls. The prizewinning guess is sometimes exact; more often it is just close. Other answers range from ridiculously low to outrageously high. The grossly wrong answers occur because many persons lack an understanding of what numbers mean. Even though they can say a number like 3698, they have no notion of what that quantity of gum balls looks like. An answer of 392 or 10,348 is perfectly all right with them.

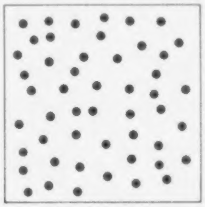
First graders use beansticks, bundles of tongue depressors, and commercial materials like Cuisenaire rods and blocks to conceptualize the meanings of relatively small numbers. A computer printout showing several hundred, several thousand, or even a million dollar signs is an impressive way to show a large quantity of something. Once primary-grade children understand the meanings of numbers, they should have periodic experiences in estimating quantities.

Activities such as the following give children practice in estimating quantities:

ACTIVITY 5-5

Guessing Dots

Show children a card and ask them to guess how many dots are on it. Allow time for them to make their guesses. Accept all responses.



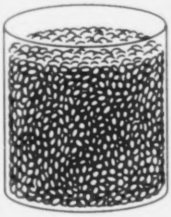
Discuss strategies used by those whose guesses were close. Include the following strategies if they are not given by the children:

☐ Count the dots across the top (or bottom) and down one side. There are about 7 each way. There are about 49 (or 50) dots on the card.
☐ Fold the card so that only one-fourth of it shows. Count the dots on one-fourth of the card. Multiply this answer by four.

Repeat with larger cards containing more dots.

The following activity is a good one for older children:

ACTIVITY 5-6 Beans in the Jar



Display a jarful of lima beans on a table. Leave a ruler, a small paper cup, a tablespoon, and the bottom part of an egg carton next to the jar. Give children time during the next day or two to guess how many beans are in the jar. Tell them they may use any strategy they wish short of removing and counting the beans.

Accept and list guesses; then discuss strategies, including the following:

☐ Measure the depth of the beans in the jar. One layer is about one centimeter deep and contains ___ beans. The beans are ___ centimeters deep, so there are about ___ beans in the jar.
☐ There are an average of ___ beans in one paper cup. (Count beans in 3 or 4 cups and determine their average.) There appear to be about ___ cups of beans in the jar, so there are about ___ beans.
☐ There are about ___ beans in one tablespoon. Pour 12 tablespoonsful into the egg carton, 1 spoonful per section. There are about ___ beans in one carton and about ___ cartons of beans in the jar. So there are about ___ beans in the jar.

Count the beans. (Can you devise a way to do this so that you reinforce children's understanding of place value and large numbers?)

Discuss the strategies in terms of whether one is more useful than another or whether they are equally useful. Under what circumstances might one strategy be better than another? (Consider the different sizes of objects that might be put in the jar.)

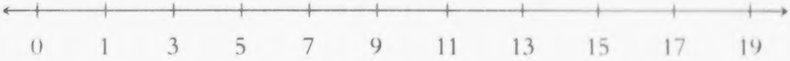
ANSWERS TO PROBLEMS

PROBLEM 5-1 T-Shirt Patterns

There are no specific answers for this problem. Possible answers for the three-letter patterns are: red-white-blue-yellow-. . . ; white-red-white-blue-. . . ; and blue-red-red-white-. . . .

PROBLEM 5-2 Arithmetic Patterns

The hundredth term in each arithmetic sequence can be determined by continuing the sequence, which is probably what children will do. It is a slow process with paper and pencil; a calculator is easier and faster. A generalized rule can be determined, however, and then used to find any term in a sequence. A number line that shows the odd numbers, beginning with 1, and a number pattern will help you understand the rule. The number line shows that each number is 2 greater than the preceding number.



$$
\begin{aligned}
1 + (0 \times 2) &= 1\\
1 + (1 \times 2) &= 3\\
1 + (2 \times 2) &= 5\\
1 + (3 \times 2) &= 7\\
1 + (4 \times 2) &= 9\\
&\vdots
\end{aligned}
$$

In the pattern, 1 is shown as $1 + (0 \times 2) = 1$. The "2" represents the difference between each number, while the "0" shows that 2 was not added to the 1. Three is shown as $1 + (1 \times 2)$ because it is one 2 greater than 1. Five is shown as $1 + (2 \times 2)$ because it is two 2s greater than one. Each successive number is shown as $1 + (3 \times 2)$; $1 + (4 \times 2)$; and so on. When you begin with zero, the hundredth whole number is 99. Thus the hundredth number in the sequence of odd numbers must be $1 + (99 \times 2)$, or 199.

Determine what the constant is in each of the sequences 4, 7, 10, 13, . . . , 3, 6, 9, 12, . . . , and 3, 9, 15, 21, ... Then use the rule to verify that the hundredth numbers for these arithmetic sequences are 301, 300, and 597, respectively.

PROBLEM 5-3 Triangular Numbers

This sequence lists the set of triangular numbers:

$$1, 3, 6, 10, 15, \ldots$$

Each triangular number greater than 1 is the sum of two or more consecutive whole numbers, beginning with 1:

$$\begin{aligned}&1 = 1\\&1 + 2 = 3\\&1 + 2 + 3 = 6\\&1 + 2 + 3 + 4 = 10\\&1 + 2 + 3 + 4 + 5 = 15\end{aligned}$$

and so on.

The number of dots in a triangle is the sum of the number of consecutive whole numbers, beginning with 1, named by the number of dots along one side. The number of dots in a triangle with 8 dots along one side is the sum of $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8$, or 36.

A rule for finding the number of dots in an n-dots-to-a-side triangle is $n(n + 1) \div 2$.

PROBLEM 5-4 The Towers of Hanoi

This table shows the moves for 1, 2, 3, 4, and 5 rings:

Number of rings	Number of moves	Differences between number of moves
1	1	0
2	3	2
3	7	4
4	15	8
5	31	16

The numbers in the third column increase as successive powers of 2: $2^1 = 2$, $2^2 = 4$, $2^3 = 8$, $2^4 = 16$, ... A comparison of the numbers in columns two and three shows that each number in column two is one less than twice the number to its right in column three.

A rule for determining the moves required to transfer n rings is $2^n - 1$, where n indicates the number of rings.

PROBLEM 5-5

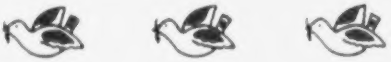
A Mystery Sequence

There is no constant difference between numbers in this sequence, so a general rule does not apply. The sequence is the set of numbers that are successors of prime numbers.

PROBLEM 5-6

How Many Birds?

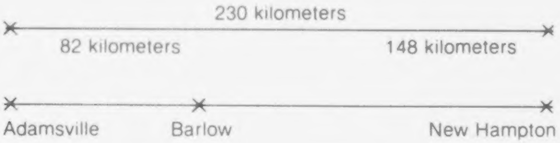
A picture like the one below verifies that 3 birds are the smallest number of birds that satisfies the conditions of this problem.



PROBLEM 5-7

Leaving Adamsville

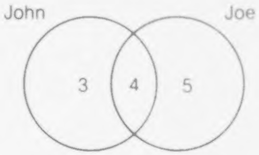
The road sign gives the information needed to fill in a diagram. The diagram shows that it is 148 kilometers from Barlow to New Hampton. Half that distance is 74 kilometers. Add 74 to 82 to determine the distance to go from Adamsville to be halfway between the two towns.



PROBLEM 5-8

The Dog Kennel

Use 2 circles in your Venn diagram, because there are 2 boys. The *3* in John's circle represents the 3 dogs he owns alone, and the *4* represents the dogs they own jointly. The *5* in Joe's circle is the difference between 12 and (3 + 4) and represents the dogs Joe owns alone.

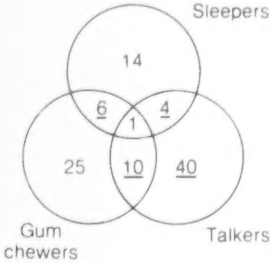


PROBLEM 5-9

The Math Teacher

Use a three-circle diagram. First fill in the information you know for certain (underlined in the diagram). Once this information is in place, use it to determine the unknown information. The known information accounts for

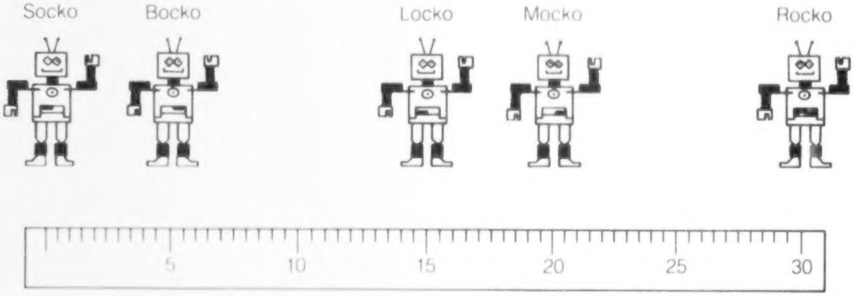
54 of the 55 talkers (40 + 10 + 4 = 54). This tells you that there can be but 1 student who talked, chewed gum, and slept. Once you know this, you can determine the number of students who only sleep or only chew gum.


Sleepers
14
6
4
1
25
10
40
Gum chewers
Talkers

The total number of students is 100 (40 talkers only, 14 sleepers only, 25 chewers only, 10 talkers–chewers, 6 sleepers–chewers, 4 talkers–sleepers, and 1 sleeper–talker–chewer).

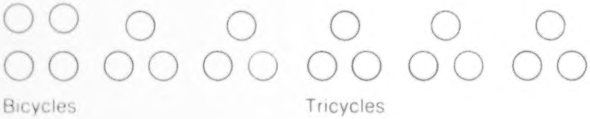
PROBLEM 5-10 A Line of Robots

Five plastic robots lined up beside a ruler show their walking order.


Socko
Bocko
Locko
Mocko
Rocko
5
10
15
20
25
30

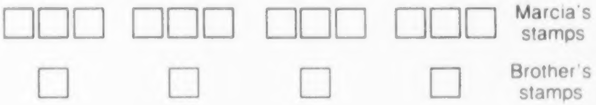
PROBLEM 5-11 Cycle Riders

Use different arrangements of 2 and 3 disks as wheels until you have the answer. These "wheels" show that 2 bicycles and 5 tricycles are ridden by the 7 riders.


Bicycles
Tricycles

PROBLEM 5-12 Stamp Collectors

The problem tells you that Marcia has three times as many stamps as her brother. Use squares of paper to show this ratio. The model shows that if Marcia has 12 stamps, her brother has 4 stamps. When each child has this many stamps, the brother can match Marcia by getting 8 more stamps.

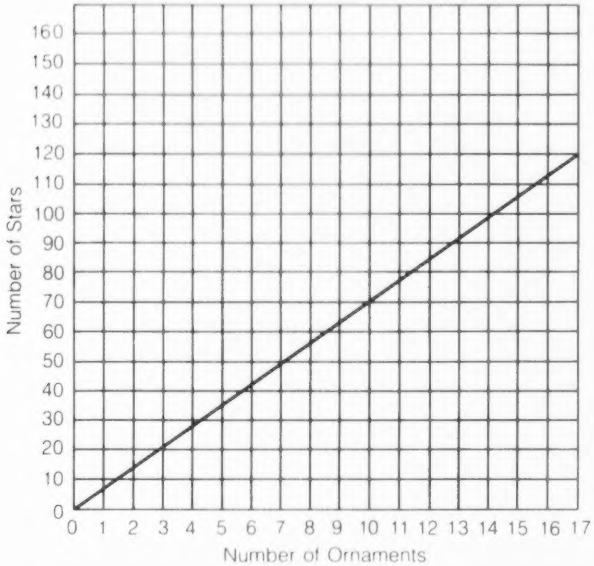


PROBLEM 5-13 Our Favorite Ice Cream Flavors

The end result of this activity is a simple bar graph. It will have a horizontal base line and columns (bars) composed of colored paper squares. Each column represents a flavor of ice cream. The number of squares in each column depends upon the number of children who chose the ice cream represented by that color.

PROBLEM 5-14 Christmas Ornaments

Both a table and a graph can be used to determine the number of stars that are needed for 16 ornaments.



Number of Stars	1	2	3	4	15	16	17
Number of Ornaments	7	14	21	28			

PROBLEM 5-15 Two Unique Rectangles

The strategy here is to use a series of tables showing length, width, perimeter, and area for rectangles.

Start with a table that shows a one-by-one rectangle and increases a pair of sides by units of one:

Length	Width	Perimeter	Area
1	1	4	1
2	1	6	2
3	1	8	3

Will this table produce a solution? Why?

Make another table.

Length	Width	Perimeter	Area
2	2	8	4
3	2	10	6
4	2	12	8

Will this table produce a solution? Why?

Your next table begins with a three-by-three rectangle. Does it yield a solution?

Length	Width	Perimeter	Area
3	3	12	9
4	3	14	12
5	3	16	15
6	3	→18	→18
7	3	20	21
8	3	22	24

We have one answer. The numbers beyond the matched pair will not yield another answer. Why?

The next table looks like this:

Length	Width	Perimeter	Area
4	4	→16	→16
5	4	18	20

This table yields the second answer.

PROBLEM 5-16 Buying Apples

It may be helpful to use a table in which to organize your guesses. For example, let the first guess be 3 big apples and 4 smaller apples, and calculate the cost.

Big Apples (50¢ each)		Smaller Apples (40¢ each)		Total	
Number	*Cost*	*Number*	*Cost*	*Number*	*Cost*
3	$1.50	4	$1.60	7	$3.10
3	1.50	3	1.20	6	2.70
4	2.00	3	1.20	7	3.20
4	2.00	2	.80	6	2.80

The first guess was not correct; it yielded a sum greater than three dollars. The second and third guesses were not correct either, though each one yielded useful information. The total cost in the third guess was twenty cents more than three dollars, which was the amount of money that Jamie gave for the apple purchase. Since 1 smaller apple cost forty cents, the next guess can be refined by omitting 1 smaller apple and reducing the total price by forty cents, The cost is $2.80, so twenty cents in change will be given to Jamie.

PROBLEM 5-17 The Ice Cream Parlor

Make a table in which the topping flavors are listed down the side and the ice cream flavors across the top. Show each ice cream-topping combination as ordered pairs in the table. There are 24 ordered pairs, or different sundaes, so there are enough for 19 children.

	Ice cream			
Topping	***Vanilla***	***Coffee***	***Banana***	***Walnut***
Marshmallow	M–V	M–C	M–B	M–W
Cherry	Ch–V	Ch–C	Ch–B	Ch–W
Chocolate	Choc–V	Choc–C	Choc–B	Choc–W
Strawberry	S–V	S–C	S–B	S–W
Caramel	Car–V	Car–C	Car–B	Car–W
Raspberry	R–V	R–C	R–B	R–W

PROBLEM 5-18 Buying Fertilizer

You can systematically account for the weight and cost of each combination of bags that will yield at least 17 pounds of fertilizer. (It is not necessary to record all possibilities; you need list only the combinations that give at least 17 pounds of fertilizer.) This table shows that the best price actually gives the customer more fertilizer than the second-best price.

Number of 5-lb bags	Cost ($)	Number of 3-lb bags	Cost ($)	Total weight (lb)	Total cost ($)
4	13.00			20	13.00
3	9.75	1	2.29	18	12.04
2	6.50	3	6.87	19	13.37
1	3.25	4	9.16	17	12.41
		6	13.74	18	13.74

PROBLEM 5-19 Handshakes

Role-play the situation with 5 friends to show that there are a total of 15 handshakes. If you represent the people by letters—A, B, C, D, E and F—you can also make a table to determine the number of handshakes. A shakes hands with each of the other 5 people for a total of 5 handshakes. B shakes hands with C, D, E, and F, giving 4 more handshakes. The process continues until each person has shaken hands with every other person. The number of pairs in the table is 15, as shown:

AB, AC, AD, AE, AF
BC, BD, BE, BF
CD, CE, CF
DE, DF
EF

PROBLEM 5-20 Horse Trading

You and a friend can use play money to role-play this situation. You will find that the horse-trading lady made twenty dollars on her transactions. She originally spent sixty dollars of her own money. The first sale gave her a profit of ten dollars. When she repurchased the horse, she spent the original sixty dollars, her ten-dollar-profit from the first sale, and another ten dollars of her own money for a total of eighty dollars. Only seventy dollars of the repurchase price was her own money, so when she sold the horse a second time, she had a net profit of twenty dollars.

PROBLEM 5-21 Airplane Trip

The commutative property for addition permits you to list the addends in an addition sentence in any order you choose, so there is no one correct sentence for this problem. Any addition sentence that includes all 4 addends is correct.

PROBLEM 5-22 Buying Groceries

Again, there is no one correct sentence. The sentence that children are likely to use is this one: $1.26 + (2 \times .59) + (3 \times .33) = \square$, which lists the numbers in the same order as in the word problem.

PROBLEM 5-23 Measuring Shadows

Answers will vary for this problem. One way to set up the proportion expression for this problem is: $a/b = c/d$. The letter a represents your height; the letter b the length of your shadow; and the letter d the length of the object's shadow; c will be the height of the object. For example, if you are 135 centimeters tall, your shadow is 195 centimeters long, and the object's shadow is 365 centimeters long, the height of the object is approximately 253 centimeters, or 2.53 meters.

$$\frac{135}{195} \qquad \frac{c}{365}$$

$$195c = 135 \times 365, \quad 195c = 49\,275$$

$$49\,275 \div 195 = 253, \quad c = 253$$

PROBLEM 5-24 At the Five and Dime

The problem says I bought as many items as the number of cents each item cost. The number of items equals the number of cents, so $n \times n =$ total cost. I must determine the number that, when multiplied by itself, equals 225. This is the square root of 225, or 15. A calculator with a square-root key is helpful here.

PROBLEM 5-25 Wine and Cork

Be careful with this one. At first glance it appears that the wine cost \$9.00 and the bottle and cork cost \$1.00. But \$9.00 is not \$9.00 more than \$1.00. A guess-and-check strategy can now be used, if necessary, to check other combinations. The wine's value is \$9.50, and the cork and bottle are worth 50 cents.

PROBLEM 5-26 The Farmer's Trees

People have a tendency to want to arrange the trees in an array pattern. There is no way this can be done with only 10 trees and still meet the other conditions of this problem. Think of other arrangements that can be used. Consider an arrangement that has a given tree in more than one row. Does the familiar star that teachers commonly mark on children's papers give a clue?

PROBLEM 5-27

Checkers

Nothing in the problem says that the 2 players are playing each other!

PROBLEM 5-28

Triangles

You need to think beyond just laying the 6 pencils on a two-dimensional plane. Rather, think of three dimensions. Early Egyptians did, and look at the pyramids they were able to build!

Summary

Problem solving is described in this book as the set of actions a person takes when faced with a problem for which an immediate solution is not apparent. For a problem to exist there must be both a solution and a commitment on the part of an individual to find a solution. A person must have a repertoire of mathematical problem-solving strategies and skills in order to be a successful problem solver. The following problem-solving strategies are among those elementary-school children can learn: Look for and use a pattern; draw a picture or a diagram; use a model; collect data and organize it in a table, graph, or chart; act out the conditions of the problem; account for all possibilities; make educated guesses and check them out; write mathematical sentences; break a problem into parts; restate the problem; identify the wanted-given information; and break set, or change the point of view with which the problem is seen.

George Polya explained a four-step problem-solving process that has been adapted for elementary-school children. The four steps are: Develop an understanding of the problem, devise a plan for solving the problem, carry out the plan, and look back to evaluate and modify, if necessary, the plan and solution.

Children learn strategies and the four-step problem-solving process through work with both real and hypothetical problem situations. Real-world problem situations are preferred over hypothetical ones because they deal with problems that are more meaningful to children. Problem situations are simplified for primary-grade children and become more complex as children progress through the grades. Textbook and problem-card problems and those generated by the children themselves provide opportunities for vicarious problem solving to supplement real-world situations and provide practice with strategies. Written vicarious problems require special attention. Procedures used to teach vocabulary in reading lessons should be used to teach vocabulary in mathematics. A clue-word strategy for dealing with written problems can be misleading in some situations and is applicable in only limited cases; it should be avoided. Children's own problems have an advantage over published problems because their settings and vocabulary are familiar. Problems with too much or too little information and problems without questions should be analyzed and revised so they are meaningful.

Estimation is frequently used as problems are solved. Sometimes estimates are made to determine the reasonableness of an answer; at other times an estimated answer is sufficient as the solution to a problem. Children need frequent opportunities to make estimates of quantities of objects, measurements, and answers for operations. The ability to round off numbers is a skill frequently used when estimating answers. Number lines and calculators can be used to develop skill in rounding off numbers.

Study Questions and Activities

1. Examine a primary-grade and an intermediate-grade book from a contemporary elementary-school mathematics series to determine which of the 12 problem-solving strategies are included. Does the teacher's manual or textbook contain activities for developing specific strategies? Are word problems and other problem-related materials organized to introduce and develop strategies? Based on your examination, what (if anything) would a teacher need to do to supplement the textbook material to give children a comprehensive problem-solving program?
2. Look at the examples in this chapter. Using them as a model, create alternative problems of your own for 5 strategies. You may want to refer to Randall Charles's article (see For Further Reading section) on "Writing Variations of a Problem." Then use other published resources to find alternative problems for 5 more strategies, noting the sources. Indicate the grade range for which you think each alternative problem is appropriate. Save the problems for your teaching file.
3. Is problem solving just a slogan? Kilpatrick (see For Further Reading section) is fearful that problem solving will be pushed for awhile, then dropped. Write a paragraph giving your response to his concern. List things you can do as a teacher to prevent problem solving from being a temporary part of the mathematics curriculum.
4. List 3 classroom situations that could be used as a basis for realistic problem solving. How would you organize those situations so that children would achieve the most experience with problem solving?
5. Some teachers post a "brainteaser" of the day or week on the bulletin board or chalkboard for children to work on. Talk to other teachers who use mathematical puzzles to find out more about this practice. Ask

questions such as: What kinds of puzzles are used? What resources are used to find the puzzles? What are the responses of children to the puzzles? Do certain children regularly work on the puzzles while others do not? What characteristics differentiate those who solve puzzles from those who don't? Does the teacher follow up and use the puzzles to help children develop their problem-solving abilities? What are the strengths and potential problems of this practice?

6. Read the letter by Krulik and Rudnick (see For Further Reading section) about the dangers of emphasizing key words in written word problems. They give several examples of ways in which key words mislead children. Write 3 or 4 original problems in which the key words might interfere with solving a problem. Select one of the 12 problem-solving strategies to replace the key-word approach as a means of solving each of your problems.

7. Use the software evaluation form (Educational Courseware Evaluation) in Chapter 4 (or a similar form) to evaluate the *Whatsit Corporation* or *Math Shop* or another simulation program. What strategies from this chapter can be used with the simulation program? Can Polya's four-step process be used as you work with the program? Discuss briefly what contributions you think the program makes to a child's problem-solving skills.

For Further Reading

Arithmetic Teacher, XXIX, No. 6 (February 1982). This issue contains ten articles about problem solving. There are both how-to and opinion articles.

Barson, Alan. "And the Last One Loses!" *Arithmetic Teacher*, XXXIII, No. 1 (September 1985), pp. 35–37. Suggests several strategy games which are won by the last move.

Bartalo, Donald B. "Calculators and Problem-Solving Instruction: They Were Made for Each Other," *Arithmetic Teacher*, XXX, No. 5 (January 1983), pp. 18–21. Describes a classroom experience for enhancing problem solving with the calculator.

Brickman, Mary. "Food for Math," *Arithmetic Teacher*, XXXIV, No. 1 (September 1986), pp. 48–49. A classroom grocery provides many opportunities for problem solving.

Bush, William S., and Ann Fiala. "Problem Stories: A New Twist on Problem Posing," *Arithmetic Teacher*, XXXIV, No. 4 (December 1986), pp. 6–9. Nonroutine problem stories can provide motivation and unusual opportunities for creative thinking.

Charles, Randall I. "Evaluation and Problem Solving," *Arithmetic Teacher*, XXX, No. 5 (January 1983), pp. 6–7, 54. Suggests ways of evaluating problem solving other than looking for the right answer.

———. "Get the Most Out of 'Word Problems'," *Arithmetic Teacher*, XXIX, No. 3 (November 1981), pp. 39–40. Most textbook word problems illustrate applications of mathematics topics. Charles describes ways to make these same problems fulfill the role of developing mathematical concepts and improving problem-solving skills.

———. "Writing Variations of a Problem," *Arithmetic Teacher*, XXXIII, No. 9 (May 1986), pp. 26–27. Tips for extending problems by changing them or having students change them.

Davidson, James E. "The Language Experience Approach to Story Problems," *Arithmetic Teacher*, XXV, No. 1 (October 1977), p. 28. Having children write and solve their own story problems based on personal experience helps avoid some of the pitfalls of textbook problems—low reading comprehension, lack of experience with the problem situations, unfamiliarity with the language of the situations, and irrelevancy of the problems.

DeVault, M. Vere. "Doing Mathematics is Problem Solving," *Arithmetic Teacher*, XXVIII, No. 8 (April 1981), pp. 40–43. There are four types of problem-solving experiences in a good mathematics curriculum. Each type has a definite contribution to make to children's problem-solving abilities.

Greenes, Carole E., and Linda Schulman. "Developing Problem-Solving Ability with Multiple-Condition Problems," *Arithmetic Teacher*, XXX, No. 2 (October 1982), pp. 18–21. Polya's four-step problem-solving model is applied to problems having more than one condition. Guess-the-number problems are used as examples for discussing procedures for all grades.

Hope, Jack A. "Estimation and Mental Computation," *Arithmetic Teacher*, XXXIV, No. 3 (November 1986), pp. 16–17. Hints for mental multiplication using aliquot parts.

Hosmer, Patricia C. "Students Can Write Their Own Problems," *Arithmetic Teacher*, XXXIV, No. 4 (December 1986), pp. 10–11. Students can invent their own problems as a classroom challenge.

Johnson, David C. "Teaching Estimation and Reasonableness of Results," *Arithmetic Teacher*, XXVII, No. 1 (September 1979), pp. 34–35. Johnson makes an appeal for more formal treatment of estimation and checking for "reasonableness of results." He discusses ways to do this.

Jones, Billie M. "Put Your Children in the Picture for Better Problem Solving," *Arithmetic Teacher*, XXX, No. 8 (April 1983), pp. 30–33. Problems based on information contained in the *Guinness Book of World Records* and the mathematical and reading skills required to solve them are presented. Five different problems are discussed.

Kilpatrick, Jeremy. "Stop the Bandwagon, I Want Off," *Arithmetic Teacher*, XXVIII, No. 8 (April 1981), p. 2. Kilpatrick pleads for substance to the slogan "Problem solving should be the focus of the curriculum." Otherwise it will be something a "teacher tried, but didn't like," as with so many other educational fads.

Krulik, Stephen. "Problem Solving: Some Considerations," *Arithmetic Teacher*, XXV, No. 3 (December 1977), pp. 51–52. Presents and elaborates on 3 criteria for identifying a "problem." Describes 7 ways to help children become better problem solvers; some are accompanied by examples.

Krulik, Stephen, and Jesse Rudnick. "Categorize Carefully," *Arithmetic Teacher*, XXXIV, No. 8 (April 1986), p. 5. A short letter to the editor points out the pitfalls of using "key" words in solving problems.

Lee, Kil S. "Guiding Young Children in Successful Problem Solving," *Arithmetic Teacher*, XXIX, No. 5 (January 1982), pp. 15–17. Three problem situations and ways children can tackle each one are discussed. Polya's model is the basis for children's processes.

Moses, Barbara. "Individual Differences in Problem Solving," *Arithmetic Teacher*, XXX, No. 4 (December 1982), pp. 10–14. Children's varying levels of understanding and skills need to be considered during problem-solving activities. Examples of materials and procedures for developing understanding and skills are illustrated and discussed.

Muller, Adelyn C., and Ray Kurtz. "Students Like Personalized Word Problems," *Arithmetic Teacher*, XXVIII, No. 9 (May 1981), pp. 13–14. Word problems that relate to each child are more motivating than the general word problems in textbooks. Ways to teach children to write their own problems are presented.

Polya, George. *How to Solve It* (Garden City, NY: Doubleday, 1957). Problems are solved by a process called *heuristic reasoning*, in which a systematic investigation of a problem is made to determine a solution. A four-step process is recommended by Polya.

Reys, Robert E. "Testing Mental-Computation Skills," *Arithmetic Teacher*, XXXIII, No. 3 (November 1985), pp. 14–16. Suggests ways of evaluating mental computational skills.

Schoen, Harold L., and Marilyn J. Zweng, eds. *Estimation and Mental Computation, 1986 Yearbook* (Reston, Va: National Council of Teachers of Mathematics, 1986). Rationale and activities for estimation and mental computation.

Shaw, Jean. "Newspapers Add Spark to Mathematics Activities," *Arithmetic Teacher*, XXXI, No. 8 (April 1984), pp. 8–13. Problems can be found on every page of the daily newspaper.

Suydam, Marilyn N. "Estimation and Computation," *Arithmetic Teacher*, XXXII, No. 4 (December 1984), p. 35. Summarizes characteristics of children who use effective estimation strategies.

Trafton, Paul R. "Estimation and Mental Arithmetic: Important Components of Computation," *Developing Computational Skills, 1978 Yearbook* (Reston, Va.: National Council of Teachers of Mathematics, 1978), pp. 196–213. Explains and illustrates the uses of estimation in problem solving and computation, processes for developing estimation skills, and ways to develop mental arithmetic skills.

Walter, Marion. "Frame Geometry: An Example in Posing and Solving Problems," *Arithmetic Teacher*, XXVIII, No. 2 (October 1980), pp. 16–18. A picture-frame-making problem is posed. How many different frames can be made? Strategies for this and similar problems are considered.

Willcutt, Bob. " 'Stamp Collector's Nightmare,' Phase Two," *Arithmetic Teacher*, XXXI, No. 1 (September 1983), pp. 43–46. Many strategy problems develop from placement of different value "stamps" in a grid.

Wirtz, Robert W., and Emily Kahn. "Another Look at Applications in Elementary School Mathematics," *Arithmetic Teacher*, XXX, No. 1 (September 1982), pp. 21–25. A distinction is made between application and problem solving. Children's personalized problems are the focus of this article.

Part 2

Procedures and Materials

Chapter

6

Teaching the Foundations of Numeration

In Chapter 6, you will read about

1. activities to develop mathematical concepts in kindergarten and primary grades, with discrete and continuous materials emphasizing classification, patterns, order and sequence, and spatial relationships
2. considerations in choosing computer software for young children
3. five principles that describe how children learn to count
4. concrete and pictorial objects to develop understanding of numbers 0–9
5. relating sets and the number line to numerals and providing practice in numeral writing
6. introducing ordinal numbers in classroom situations
7. music, finger-plays, poems, and stories that enhance children's understanding of mathematics concepts
8. difficulties that children encounter in early number work and activities to help them

There was a time in the past when formal instruction in mathematics was delayed in many schools until at least third grade. Between 1920 and 1950 many educators believed that until children reached their eighth birthday they were too immature to understand number and other mathematical concepts. If mathematics concepts were presented to children, it was through incidental, unplanned activities. Some teachers were adept at using chance opportunities to give their children experiences with numbers and other mathematics concepts. However, many were unable to capitalize on such opportunities, or they ignored them, and their children learned nothing about mathematics in school.

Today few educators believe that children below grade three should be denied systematic instruction in mathematics. Children's work commonly begins in grade one, and in many schools preschool and kindergarten children have planned mathematics activities.

EARLY MATHEMATICS EXPERIENCES

Most children understand some mathematical concepts at an early age. It is difficult to identify any particular moment in a child's life when the first numerical or quantitative concepts are acquired. Understanding of mathematical concepts begins imperceptibly and grows so gradually it is almost unnoticeable. It has been observed that children under two years of age have acquired some idea of *more*, usually in connection with food, such as taking the larger cookie or piece of candy when given a free choice.[1] Rochel Gelman and C. R. Gallistel report studies that show that two- and three-year-old children understand the process of counting objects. Although the children were not accurate, they showed that they were aware of the principles involved.[2]

The mathematical understandings children develop in preschool years depend upon their experiences. Many pick up simple number skills through both deliberate and incidental activities. For example, many children are taught to say number words in sequence. Usually this is done without regard to the meaning of the numbers. Such counting is referred to as rote counting. Though this skill is not particularly valuable in itself, it is a foundation upon which later work can be built. A deliberate effort to teach meaningful counting occurs when an adult encourages children to count tableware pieces as they help set the table. Children who play at game boards on which moves are determined by dice or spinners learn to count spots on dice and read spinner numerals, as well as to count moves around the board.

---

[1] George E. Hollister and Agnes G. Gunderson, *Teaching Mathematics in the Primary Grades* (Boston: D. C. Heath, 1964), pp. 12–13.

[2] Rochel Gelman and C. R. Gallistel, *The Child's Understanding of Number* (Cambridge, Mass.: Harvard University Press, 1978), pp. 131–135.

Many simple skills are acquired incidentally through play. Not only are these experiences fun but also they help children acquire vocabularies and background upon which later learning is based. In the same way that language is learned spontaneously in play, so are mathematics skills learned. Children use counting processes; fit shapes into forms; make symmetric patterns; build block structures; crawl over, under, and around objects; and engage in other activities with a mathematics orientation.

PRENUMBER EXPERIENCES

Informal, play-type experiences should continue in preschool and kindergarten classrooms and in first grades, too, depending upon the children's earlier experiences. A room should have ample teacher-made and commercial materials to help children develop a beginning understanding of these topics:

Discrete and continuous materials
Patterns
Ordering and sequencing
Spatial relationships
Sorting and classification

Free-Play Activities

a. Discrete and Continuous Materials Discrete materials are those that can be counted, such as cookies, children, toys, and dots on a piece of paper; while continuous materials are those that are measured in order to attach a numerical value to them, such as children's heights, containers of milk, and packages of rice. Free-play activities let preoperational-level children have experiences with both types of materials. Later, teacher-directed lessons give guidance to activities with discrete and continuous materials to help children see the materials' characteristics and to understand relationships, patterns, and sequences, and to develop vocabulary.

Ample opportunities should be provided in early school years for children to engage in activities like these:

ACTIVITY 6-1

Rice-pouring Center

Materials: Plastic wading pool (about 4-foot diameter and with rigid sides), plastic wash pan, 15–20 pounds of rice (rice can be colored by soaking it in food-color-dyed water for several hours and then drying thoroughly), cans and jars of various sizes up to about one quart. Rice is kept in the wash pan, which is placed in the center of the wading pool. Jars and cans are in the pool. (Even though each individual grain can be counted, rice is considered a continuous material when used in this activity.)

> Children sit around the pool to fill and empty containers and pour from one container to another.
>
> Variations: Take the equipment outdoors and fill the wash pan with water or clean sand.

This activity helps children develop an understanding of conservation of a continuous material and to learn intuitively the concepts of more and less than, equality, and ratio. These concepts are the foundations upon which, in later years, an understanding of volume is built. The activity also provides opportunities for counting as children fill a large container and pour the contents into smaller ones and count the number of cans that are filled; or as they fill a large can by repeatedly emptying the contents of a smaller can into it.

ACTIVITY 6-2

Building-block Center

Materials: Large, hollow wooden building blocks.

> Children's free-play activities with blocks result in the construction of buildings, walls with doorways, roadways, boats and airplanes, and even more fanciful structures.
>
> Variations: Blocks made from two-by-four and four-by-four pieces of wood.

This activity gives children opportunities to climb on, under, over, beside, and around the blocks. Children's understandings of spatial and size relations and patterns, as well as such terms as *under, above*, and *beside* are enhanced through play with blocks. They will also learn geometric terms like *square* and *triangle* as they play with blocks having these shapes.

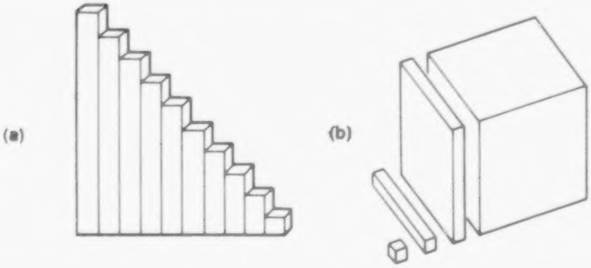
ACTIVITY 6-3

Structured Materials Center

Materials: Cuisenaire rods. (Cuisenaire rods are colored wooden blocks. The rods have a 1-centimeter cross section and lengths that are multiples of 1 centimeter. The smallest rod is a 1-centimeter cube; the longest one has a measure of 10 centimeters. There are also flats and larger cubes, which are useful for activities in which place value is involved. [See Figure 6-1].)

FIGURE 6-1

Cuisenaire rods. (a) A set of 10 rods; (b) a short and a long rod, a flat, and a large cube.



First experiences with Cuisenaire rods, or any other structured material, should be free-play activities. Children will make stair-step arrangements, towers, walls, and other imaginative structures.

During play, children will discover and describe the rod's characteristics and gain a familiarity with the materials that will serve as a basis for understanding the rods' structured properties during teacher-directed lessons.

ACTIVITY 6-4

Dressing Up

Materials: Adult clothes such as hats, gloves, jackets, boots, and shoes.

Children delight in "dressing up." When given a choice of activities, many children choose to go to the clothes center to don adult clothes.

Dressing-up activities help children learn about one-to-one matching as they button and unbutton clothes, teach them about larger and smaller as they compare sizes, and introduce them to other mathematics-related ideas.

"Math boxes" are mentioned in Chapter 1 as a source of materials for young children's manipulative activities. A math box is the size of a cigar box and contains a collection of similar objects. One box might contain red, blue, and yellow two- and four-engine plastic airplanes. Another might have plastic insects of varying sizes and colors; while a third might have two sizes of green, red, white, yellow, and brown buttons with two or four holes in them. The variety of boxes is limited only by a teacher's imagination and resources.

ACTIVITY 6-5

Math Boxes Center

> Math boxes are available so that a child can select a box and do whatever he or she wants with the contents.

Some children will sort the contents into subsets based on color, size, shape, number of holes, or another common attribute. Others will form lines or trains with their objects, with or without a pattern in the arrangement. Still others will just "play" with their materials; if there are airplanes they will "fly" them, and if there are motorcycles they will "drive" them. Accept any responsible activity, even those where "play" is all a child does. Early activities with the math boxes lay foundations for later work with one-to-one matching, counting, addition, subtraction, common fractions, and other mathematics concepts that can be represented by their contents.

Activities in which children make one-to-one matchings refine their understanding of the *as many as*, *more than*, and *fewer than* relations. Magnetic- or flannel-board materials are useful for the first two of the activities that follow. Canoes and paddles can be cut from construction paper and pictures from magazines and catalogs; attach one or more pieces of self-adhesive magnetic tape to the back of each item.[3]

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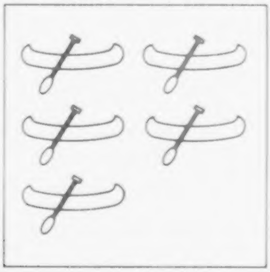
[3] Adhesive magnetic tape is available from school and business supply establishments.

ACTIVITY 6-6

Matching Canoes and Paddles

- Children match canoes and paddles.

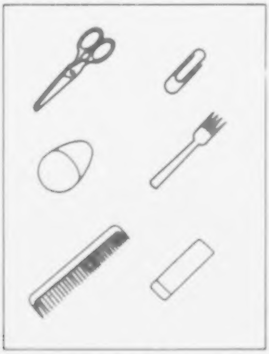


- Have the children tell whether there are *as many as*, *more than*, or *fewer than* when matchings are complete.
- Remove the paddles and hold them in one hand. Ask, "Are there as many paddles as there are canoes?"
- Variations: Use airplanes and hangers, cutout dolls and pieces of clothing, and other commonly matched items.

ACTIVITY 6-7

Matching Outlines

- Put outlines of common objects on a large sheet of heavy tagboard.

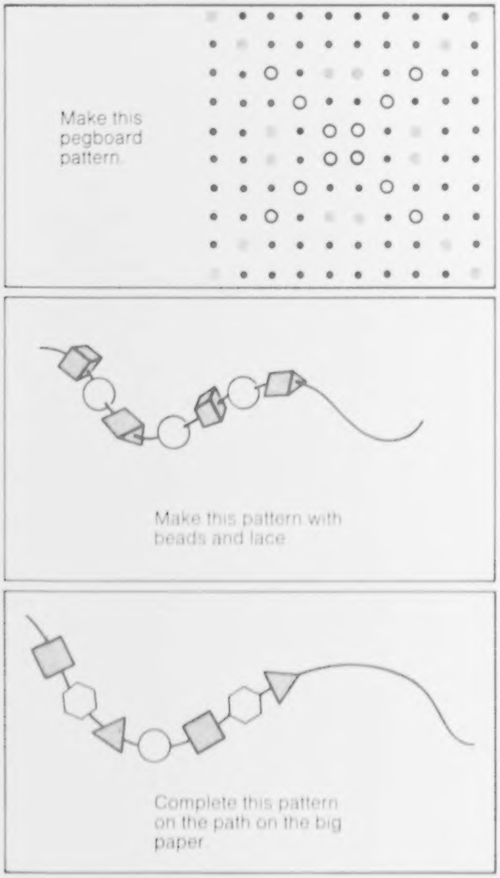


- Children match each real object with its outline.

b. Patterns Patterns abound in mathematics. When young children work with patterns involving physical objects and such things as tom-tom or cymbal beats, they sharpen their perception and develop awareness of order, sequence, shapes, and aesthetics. A pattern center can be established at a table or on the floor in a corner. Materials for a pattern center include such things as colored beads and cord, pegboard and golf tees, shapes cut from colored cardboard, pattern blocks, small colored tiles, and colored plastic disks. Directions for activities can be put on five-by-eight-inch file cards. Three cards are illustrated in Figure 6-2. The first card illustrates a pattern made with two colors of golf tees in a piece of pegboard. The pegboard has 4 spool legs that raise it above the work surface. The second card illustrates a pattern of colored beads on a cord. For the third activity, a child places shapes cut from different colors of cardboard on a line drawn on a large piece of cardboard to match the pattern shown on the problem card.

FIGURE 6-2

Samples of pattern cards


Make this pegboard pattern.
Make this pattern with beads and lace
Complete this pattern on the path on the big paper.

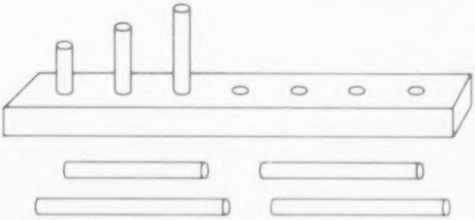
Naturally you will have cards available with other patterns for children to duplicate, and you will encourage children to make their own patterns with the materials.

c. Order and Sequence Order and sequence apply to both discrete and continuous materials. Discrete materials are counted, and their sets can be placed in order according to the number of objects. Continuous materials are measured and can be ordered according to length, height, weight, and so on. Before children learn to count and measure, however, they should have opportunities to deal with objects that can be sequenced by visual inspection. Materials for an ordering-and-sequencing center will include dowel rods, wooden blocks cut from two-by-fours, ribbon, lidded jars containing different amounts of colored water, different-sized beanbags, different-sized metal washers to be fitted over nails in a board, and other similar materials.

ACTIVITY 6-8

Ordering by Size

Use a set of ½-inch dowel rods, each a different length, and a two-by-four block with holes of equal depth. Children are to order the dowels according to length, beginning with either the shortest or the longest rod.



d. Spatial Relationships A good understanding of spatial relationships is an important characteristic of an effective problem solver. Many problems deal with relationships between and among objects and places, such as when a designed playground designates playing spaces by painted lines; or when a classroom is organized to accommodate different work spaces and arrangements of pupils' desks or tables. Individuals deal with spatial relationships when they design art projects, make a chart or graph to display data, and prepare dioramas to portray historical settings.

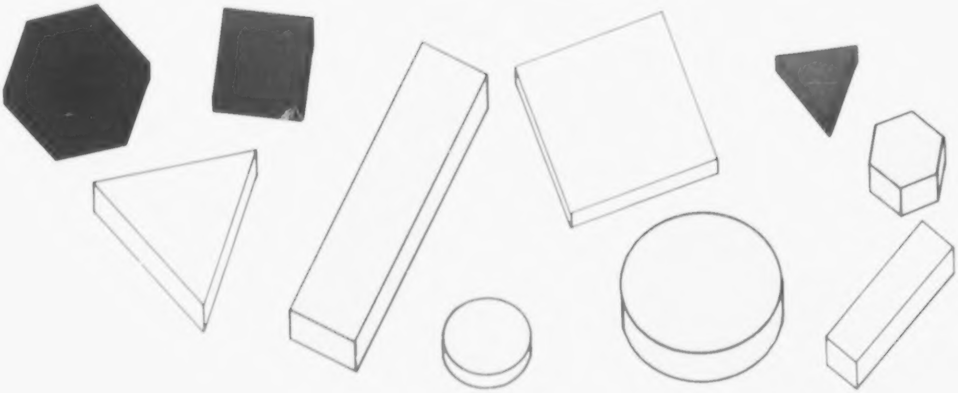
Early experiences with spatial relations and learning concepts such as *near, far, close to, under, above,* and so on, come from free-play activities in centers where children use blocks (such as in Activity 6-2) and toys (such as miniature train sets and city street scenes with cars, buildings, and trees); in classroom playhouses; and on the playground.

e. Sorting and Classifying Throughout their years in school, children sort and classify objects and ideas. Geometric objects are classified according to the number of sides and angles they possess, the shapes of their faces, and so on. Numbers are classified according to the quantities they represent; whether they are odd or even, prime or composite, are divisors or multiples of given numbers, and so on. Outside the field of mathematics, children make decisions about information they gather during a science or social studies project and prepare outlines, tables, and charts to organize their information. You can help children sort and classify concrete materials in preschool, kindergarten, and first-grade classes by using materials designed for classification activities and materials you design yourself. One set of commercial materials has 60 pieces, with 5 shapes, 3 colors, 2 sizes; and 2 thicknesses (Figure 6-3). Teacher-made sets can be objects in math boxes, packs of cards with pictures of objects, lengths of ribbons, and similar objects. Children's first experiences with an attribute set should occur in free-play situations. At first, children will choose an easily ascertained attribute, such as color or size, and separate the blocks into red, yellow, and blue or into large and small subsets. Once a child does sort a set, your question "What did you do with the blocks this time?" may cause the child to reflect upon the scheme used. Do not press those who cannot readily explain a scheme.

Sorting can be done at centers where activities like the following are available.

FIGURE 6-3

Samples of shapes in a set of commercial attribute materials



ACTIVITY 6-9

Sorting by Number of Wheels

Make a pack of cards with pictures or use models of vehicles, toys, and machines that have different numbers of wheels, such as cars, bicycles, and 1-wheeled cultivators.

Fewer	Same	More

Place 1 card or model in the middle column of the chart. Children sort the cards or models and place each one in the proper column.

ACTIVITY 6-10

Comparing Lengths

Prepare a box of long and short pieces of twine, hair ribbons, shoelaces, yarn, and so on. Children put them in the proper place on the chart.

Long
Short

Teacher-directed Activities

Following free-play activities, you will want to help children develop vocabulary, organize their thoughts about the materials, and internalize concepts. Teacher-led activities like the ones that follow will provide the direction children need.

a. Discrete and Continuous Materials As children play at a rice center (Activity 6-1), you can ask questions such as: "How many of the red cans can you fill from a full blue can?"; "Does it take more red cans to fill a green can or a blue can?"; or "Which holds more rice, a yellow can or a green can?" Similar questions can be raised as children play with blocks, structured materials, clothes, and so on. The following activity illustrates a teacher-directed lesson intended to develop vocabulary and understanding of the characteristics of objects.

ACTIVITY 6-11

Describing Objects

Use classroom objects such as chairs, tables, filing cabinet, chalk, paper towels; and objects you provide, such as swatches of cloth, pieces of sandpaper, lengths of colored dowel rods, geometric shapes, and buttons. Have children describe them by use, color, texture (feel), materials, size, shape, and so on.

Write experience stories dictated by children:
"A chair is to sit on."
"Our chairs are made of metal and plastic."
"Our chairs are red and green."
"There are big chairs and small chairs in our classroom."

---

b. Patterns The T-shirt pattern activity described in Problem 5-1 is teacher directed to give children experiences dealing with color patterns. If children use letters to represent T-shirt patterns, the activity serves as a bridge between concrete patterns and abstract number patterns that come later. Two other teacher-directed activities follow.

ACTIVITY 6-12

Listen to the Tom-Tom

Materials: A tom-tom, or similar percussion instrument, and drumstick.

Form the children in a partial circle on the floor. Sit at the open side with a tom-tom in your lap. Ask the children to listen while you beat a very simple pattern, such as a steady, unvarying *thump, thump, thump,* . . . . Have the children clap with you as you repeat the beat. Ask the children to identify a letter pattern that might be used to describe the beat. (They might suggest a steady a-a-a, ...) Ask for a volunteer to imitate your pattern on the tom-tom.

- Repeat with other simple rhythmic patterns, such as *thump-thump, thump-thump, thump-thump,* and *thump, thump-thump; thump, thump-thump; thump, thump-thump,* . . . . (Letter patterns for these might be a-a, a-a, a-a, . . . and a, b-b, a, b-b, a, b-b, . . . , respectively.)
- Have individuals think of and beat their own patterns. Have other children respond by clapping their classmates' beats and identifying a letter pattern.

ACTIVITY 6-13

Long, Short, and In-Between Toots

Materials: A flutophone for each child and yourself.

- Have children sit at their desks or in a partial circle on the floor. Sit or stand where the children can see you. Play a sequence of sounds, all of the same tone. The sequence might be long, short, long, short, ... Once children have described the sequence, repeat it so they can play with you. Identify the pattern with letters. (a, b, a, b, . . .)
- Repeat with other patterns: Long, short, medium, long, short, medium, . . . ; short, short, long, medium, short, short, long, medium, . . . ; and so on.
- Let children generate their own patterns.

---

c. Order and Sequence You will help children understand new concepts if you pay close attention to vocabulary during teacher-directed lessons. For example, when materials are ordered according to height, help children use the terms *shorter, taller,* and *in-between.* As children do the following activity, you will deal with concepts such as *smaller, larger, a little bigger, in-between, left,* and *right.*

ACTIVITY 6-14

Ordering Geometric Shapes

Materials: Sets of geometric shapes cut from colored construction paper: triangles, squares, circles, and rectangles. Each shape in a set is a different size and color. A sign that designates the shape of each set or one more shape of each type. Put 1 sign or geometric shape on each wall of the classroom.

- Give each child one of the colored shapes.
- Tell each child to locate the sign that names his or her shape and to walk

to that sign. (If the children cannot read the words, use the shapes and have them walk to shapes that match the ones they hold.) Tell the children to order themselves according to the sizes of the shapes they hold, either large to small or small to large, beginning at the left. Once the children are lined up properly, ask questions such as: "What is the color of the shape at the left end?"; "What is the color of the shape that is to the right of the yellow shape?"; or "Name the colors of the shapes, beginning with the second shape from the left."

d. Spatial Relationships An effective teacher-directed activity in this area is a game of follow-the-leader conducted on the playground. Lead the children *around* the basketball backstop, *under* the low monkey bar, and *through* the large concrete pipe. Be sure that the action words are emphasized. In-classroom activities develop spatial understandings and give practice in listening for and following directions. "Go stand *inside* the ring formed by the red Hula-Hoop." "Put the yellow block *on top of* the green block."

Two- and three-step directions should follow the simpler ones. "Go to the *front* of the room, *touch* the chalkboard, and then go stand *beside* the file cabinet."

Later, concepts of *far, near, left, right, high, low*, and so on should be developed through similar activities. Other activities dealing with spatial relations are presented on pages 240–246.

Teachers sometimes question the use of computers with young children who need many concrete experiences. The computer should not replace any of the manipulative activities suggested in this chapter. It can, however, provide a unique new interactive environment for children. Control of the computer, active participation, and the development of concepts and reasoning rather than rote memory are important. Children should be adequately prepared through work based on concrete materials before engaging in any computer activities.[4] *Juggles Rainbow*[5] is a program for young children that turns the keyboard into a command post for exploring the meanings of *left, right, above*, and *below* to give children opportunities to further develop their understanding of these relational concepts.

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[4] Steve Tipps and Lynne Mann, "Computers: A New Learning Environment for Young Children," *Dimensions* (October 1984), pp. 15–18.

[5] The Learning Company, 6493 Kaiser Drive, Fremont, CA 94555.

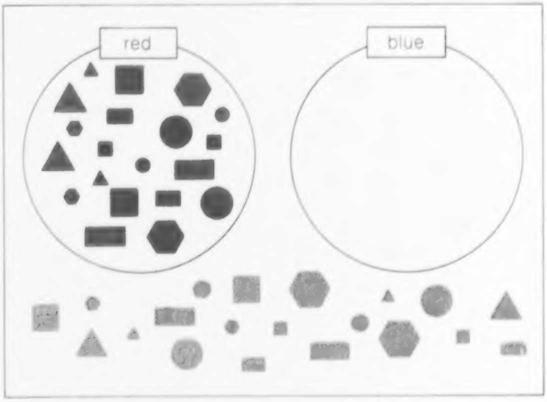
In *Stickybear Opposites*[°] children do more than watch the screen. By touching keys they change the animated scenes to demonstrate ideas such as up-down, inside-outside, and full-empty.

e. Sorting and Classifying When you observe children who are ready to move beyond free-play activities with attribute materials, introduce directed lessons. This is a good time to introduce Venn diagrams to organize the materials.

ACTIVITY 6-15

Classifying by Color

Put 2 loops of yarn on a table. Label one *blue* and the other *red*.



Direct children to put attribute blocks into each circle according to color. Ask questions: "Why is this piece here?" (as you point to a red piece); "Why are these pieces here?" (as you point to the yellow pieces outside of both circles).

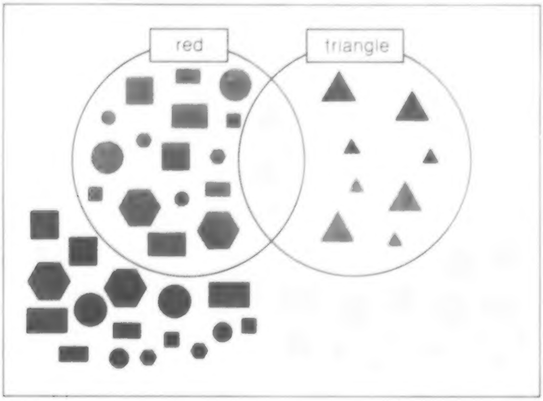
Variations: Change labels so children sort by size, shape, thickness.

[°] Weekly Reader Family Software, 245 Long Hill Rd., Middletown, CT 06457

ACTIVITY 6-16

Classifying by Color and Shape

- Put down 2 overlapping circles. Label one *red* and the other one *triangle*.



Direct children to use the labels as guides to put blocks inside the circles. Ask questions: "Why are these shapes in this circle?" (as you sweep around the red circle); "Why are these pieces inside both of the circles (as you point to the red triangles); "Why are these pieces inside neither circle?" (as you point to nonred, nontriangular pieces).

These activities for beginners in mathematics do not exhaust the things you can do to lay a firm foundation for later work. If you work with children still in the preoperational stage of mental development, you are encouraged to use some of the books listed in Appendix C for additional activities.

Two software programs that provide classification experiences are *Gertrude's Puzzle*[7] and *Moptown Parade*.[8] In *Gertrude's Puzzle* children use classification skills to determine which shapes and colors fill in a matrix. *Moptown Parade* has a group of characters that form parades when students correctly identify the characters by body shape and color. Young children cannot complete all of the activities in this program, but it has a progression of activities from easy to more difficult so children can begin with the simpler activities and move on to the more difficult ones as their knowledge of the concepts matures.

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[7] The Learning Company, 6493 Kaiser Drive, Fremont, CA 94555

[8] The Learning Company

FIRST COUNTING EXPERIENCES

Children enter kindergarten and first grade with varying number experiences and different foundation skills, including counting, so you should use the diagnostic procedures described in Chapter 3 to determine each child's counting skills. The meager backgrounds of some children may indicate that they are not yet ready to begin counting and need more prenumber activities. Other children may have advanced skills and be ready to deal with concepts such as place value as they learn about numbers greater than nine (see Chapter 7). Children who are learning to count need carefully sequenced, teacher-led activities.

Five principles apply to the process of determining the size of a set.

> The one-to-one principle involves *ticking off* the items in a collection in such a way that one and only one number is used as each item is counted.
>
> The stable-order principle means that counting numbers are arranged in a sequence that does not change.
>
> The cardinal principle gives special significance to the number named last, because not only is it associated with the last item but it also represents the total number of items in the set. This number, which tells *how many*, is the *cardinal* number of the set.
>
> The abstraction principle states that any collection of real or imagined objects can be counted.
>
> The order-irrelevance principle states that the order in which items are counted is irrelevant.[9]

An understanding of counting is built upon a child's intuitive understanding of the number *one*. There is no way to describe the meaning of *one* other than by holding up one finger, pointing to one marker on a magnetic board, or isolating some other single object in a similar way and then attaching the word *one*, and later, the numeral "1," to it. When an intuitive understanding of *one* is established, *two* can be thought of as *one-more-than one*, *three* as *one-more-than two*, and so on, until a child realizes that each number has a successor.

Gelman and Gallistel identify common errors committed by children who are learning to count:

---

[9] Rochel Gelman and C. R. Gallistel, *The Child's Understanding of Number* (Cambridge, Mass.: Harvard University Press, 1978).

- A child may make a coordination error. This occurs when the count is not started until after the first item has been touched, which results in an under-count; or when the count continues after the final item has been touched, which results in an over-count.
- A child may make omit errors. This occurs when one or more items are skipped.
- A child may make a double-count error by counting one or more items more than once.
- A child may repeatedly use an idiosyncratic counting sequence such as *one, two, three, five, seven, eight.*[10]

An awareness of these errors will alert you to watch for them as children learn to count.

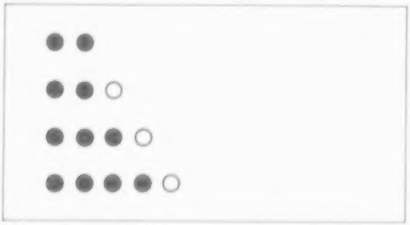
Learning to Count

Children's initial counting experiences should involve only a small number of objects. Activities like the following are recommended to avoid the errors previously mentioned.

ACTIVITY 6-17

Counting Small Groups

- Put 2 markers on a magnetic board. Ask a child to name the number of markers.



- When the number *two* has been named, direct the children's attention to other groups of two in the classroom, such as two erasers in the chalkboard rail; have them locate other sets of two.
- Arrange a second set of two on the magnetic board beneath the first set. Put another marker of another color after these two. Ask, "How many markers in this set?"
- When the number *three* has been named, have the children find groups of

---

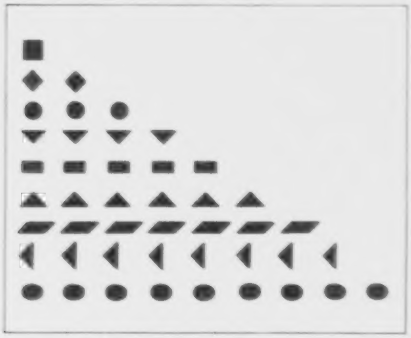
[10] Gelman and Gallistel, pp. 106–108.

three in the room. (If your room does not have several natural sets with 2 or 3 objects, prearrange some items beforehand, such as groups of pictures, books, flowers, and similar objects.)

Sets for the rest of the numbers through nine should gradually be developed and studied in a similar manner. As counting is taught, stress numerical order. Also stress the cardinal number by emphasizing it for each set you count: "One, two, *three*." After a child has counted, ask, "What is the last number you named? How many shapes (books, disks) in the set?" Figure 6-4 shows how a magnetic board or chart with geometric shapes might appear after all nine numbers have been studied.

FIGURE 6-4

Geometric shapes used on a sequence-of-sets chart to illustrate the one-more-than, or successor, pattern

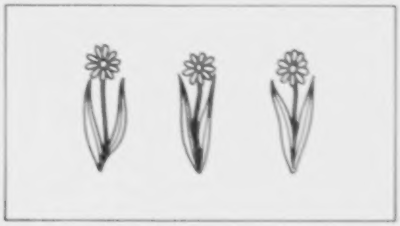


Zero requires special attention. It is not a number that children are likely to experience before school, and the set with which it is associated (that is, one with no members) is more difficult for children to comprehend than sets that contain objects. Story situations like this one are useful for introducing zero:

ACTIVITY 6-18

Learning about Zero

Put 3 felt flowers on a flannel board.



Say "These are some flowers that were growing in my garden. How many flowers are there?"

Remove 1 of the flowers and say "I went to my garden on Saturday and picked 1 of the flowers. How many were left?"

Remove the 2 flowers. "I picked the 2 flowers on Sunday. How many were left?" Children will respond with: "There aren't any." "They are all gone." "None." Accept all responses.

Show several empty containers, such as boxes with lids, covered jars, and paper bags, and ask the children to describe their contents. Some will say that they are empty. At this point introduce the word *zero* as the name for the number that tells how many flowers were left in the garden and the number of things in the containers.

Developing Number Sense

After children recognize the numerals zero through nine and can match each of the numbers with the correct number of objects, activities that develop a sense of number should be provided. Children need to be aware that the number that names the quantity of objects in 2 or more equal-sized sets is the same regardless of the objects used to depict the sets and the way the objects are arranged. The first activity that follows deals with same-sized sets that contain unlike objects. The second deals with like objects in same-sized sets that are arranged differently.[11]

ACTIVITY 6-19

The Number Seven

Materials: Pictures of animals such as penguins, rabbits, cats, and teddy bears, with magnetic tape on the back and a magnetic board.

Show 2 or 3 sets of 7 animals. Draw a loop around each set.

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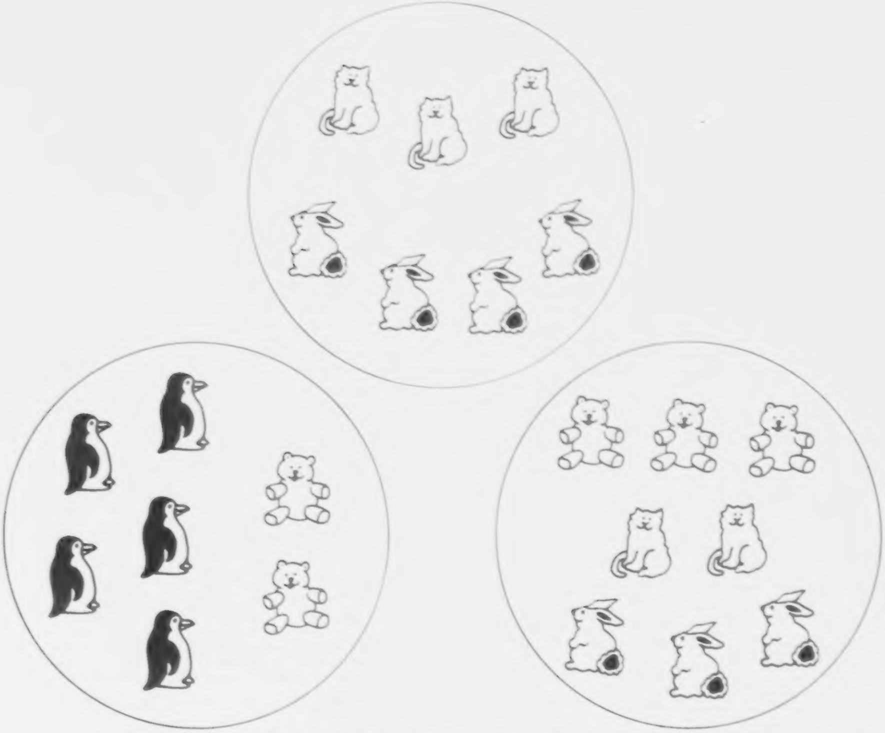
[11] These activities are adapted from Werner Liedtke, "Young Children—Small Numbers: Making Numbers Come Alive," *Arithmetic Teacher*, XXXI, No. 1 (September 1983), pp. 34–36.



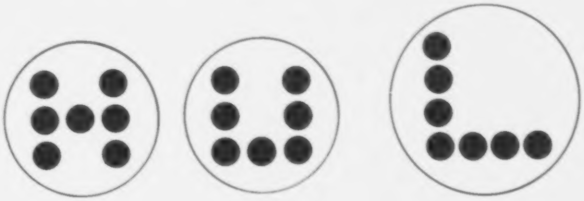
Ask the question "In what ways are these sets different?" Discuss the differences the children see.

Ask: "In what way are these sets the same?" (Each one has 7 animals in it.)

ACTIVITY 6-20 Arrangements of 7 Objects

Materials: Blue poker chips with magnetic tape on one side and a magnetic board.

Show several arrangements of 7 chips. Draw a loop around each set.



Discuss ways in which the sets are different, and a way in which they are the same. (Each set has 7 objects in it.)

Naturally you will do similar activities with other numbers less than ten until children have a clear understanding of the meaning of each number.

Clements and Callahan[12] recommend that a teacher deliberately make counting mistakes for children to catch. Errors, such as counting 1 object twice, skipping objects, or skipping a number, are the kind they recommend. Once children recognize an error, ask questions such as: "What happened?"; "Why?" Then say, "Raise your hand if you can come up and show me how I should have counted the chips." Explaining another's error helps children become more aware of their own counting processes.

Learning to Read and Write Numerals

The initial steps in learning to count take place before children are taught to write numerals. It is not necessary, however, to teach the meaning of all the numbers 0 through 9 before beginning instruction in reading them. After children have learned the meaning of *two*, *three*, and *four*, the numerals "2," "3," and "4" can be introduced.

Children can match numerals and their sets through exercises with markers. Magnetic or felt numerals are particularly useful because children can put a numeral alongside a set without having to write. Placing a numeral with its set and placing the correct number of markers next to a numeral are the most basic and simplest exercises.

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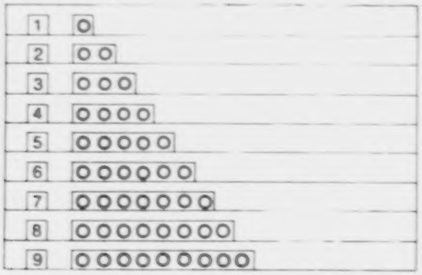
[12] Douglas H. Clements and Leroy G. Callahan, "Number or Prenumber Foundational Experiences for Young Children. Must We Choose?" *Arithmetic Teacher*, XXXI, No. 3 (November 1983), p. 35.

ACTIVITY 6-21

Matching Numeral and Set Cards

Put set cards in a pocket chart. Give children numeral cards, beginning with *1*, to put in front of the sets.

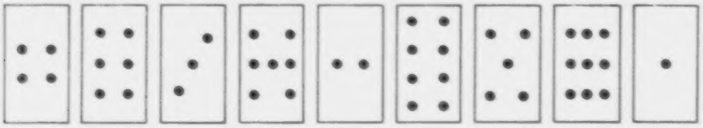


Variations: (1) Begin with *9* and reverse the order of putting the numeral cards in the chart. (2) Put the cards in the chart without regard to order. (3) Put numeral cards in the chart, and have the children put in the set cards.

ACTIVITY 6-22

Matching Domino Cards and Numerals

Give children domino cards to put in sequence, beginning with *zero*. Have children match each domino card with its numeral card. Later begin with *nine* and reverse the order.



Variation: Put a card in front of the children. Ask, "Which card goes just before this card?"; "Which card goes just after it?"; "Who can put the rest of the cards in place?"

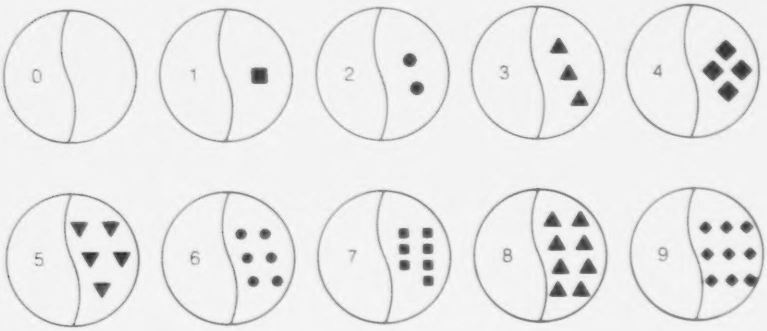
Encourage children to identify the number of a set on sight, rather than by counting each time. Most children will learn to recognize sets of four or five. Some will learn common domino patterns for larger sets. Few persons, including adults, will recognize the number of objects in randomly organized sets larger than four or five.

ACTIVITY 6-23

Matching Numeral and Set Circles

Use magnetized circles of colored cardboard containing a numeral and a set. Cut the circles so that the numeral is on one half and the set on the other half.

Put the set halves on the board so the children can match each one with the proper numeral half.



The manuscript form of writing numerals is used almost exclusively, even though most people change to the cursive form for writing words (Figure 6-5).

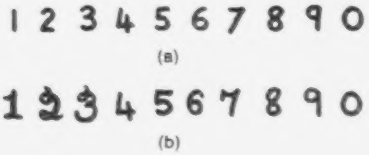


FIGURE 6-5

(a) Manuscript form for writing numerals. (b) Cursive form for writing numerals.

Write-in textbooks for the first grade include instruction pages on which children can practice writing. A lesson commonly begins with an example of a numeral. Next the numeral is made with broken lines or is rendered in light-colored ink for children to trace. Then only the starting point appears, showing the child where to begin writing the numeral. Last, there are spaces in which the child practices writing the numerals with no guidance. Figure 6-6 shows a typical exercise.

The authors of the CDA Math Program[13] advise against requiring chil-

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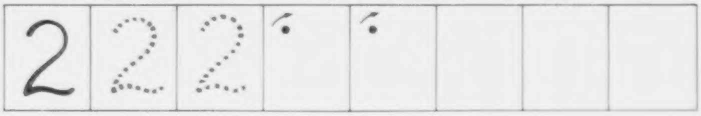
[13] Morton Botel, Robert W. Wirtz, and Max Beberman, *Memo from the Authors of CDA Math—Introducing Children to the Primary CDA Math Program* (Washington, D.C.: Curriculum Development Associates, Inc., 1973), pp. 10–11.

FIGURE 6-6

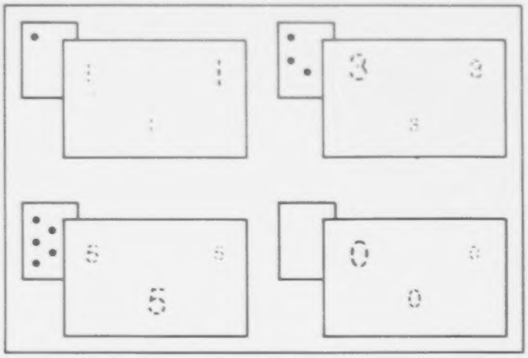
The sequence of steps for teaching children to write a numeral



dren to trace ready-made numerals. They have observed that many children constantly check their progress when they are required to trace models and become upset when they stray too far from the models. The authors recommend that children be given sheets of paper that contain 4 different numeral models. A sample of one practice page is shown in Figure 6-7. Children practice by writing the numerals in the open spaces in each numeral's box. If they wish, they can trace the models.

FIGURE 6-7

Sample numeral-writing practice page recommended by CDA Math *(used by permission)*



As children gain experience in writing numerals, most of them make numerals correctly and neatly during practice sessions. However, some are less careful when some other phase of mathematics is emphasized. Reminding a child of how a particular numeral is formed is usually all that is needed to prevent a bad habit from developing. If a chart showing the correct way to write each numeral is posted in the room, a child who makes an error can look at it to see how a particular numeral should appear. If necessary, the child can go to the chart to trace the numeral a few times with a finger before writing it on paper.

Instruction in reading number words should follow instruction in reading numerals. Even though these words are usually taught during reading lessons, they should also be taught as part of the mathematics program. Many of the exercises used for associating numerals and pattern cards can be used to learn the words and their meanings. The children match cards, numerals, and words by arranging cards in a pocket chart, as shown in Activity 6-21, or on flannel or magnetic board. You should also encourage children to use number words in their own stories.

Stickybear Numbers[14] and *Charlie Brown's* 1, 2, 3s[15] are computer programs that emphasize counting and set-numeral matching. Both programs make effective use of color and graphics.

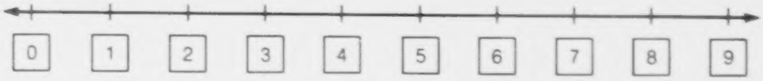
Using the Number Line to Relate Sequence and Counting

A number line is a useful device for children of all ages. Children in kindergarten and first grade can use a number line marked on the classroom floor as they learn to count. Such a line can be painted or put down with masking tape. The points should be about 14 inches apart, with the numerals that name the points printed on cards rather than on the floor to increase the number line's versatility.

ACTIVITY 6-24

Walking the Number Line I

Group children so the line runs from their left to their right.



Have a child stand on *0*. As the child takes a step from *0* to *1*, the children say *one*. Emphasize that it is the step that is counted, not the *1* card. The count continues in unison as the child moves to *9*.

Have each child walk the line while the group counts in unison or the child counts alone.

Variations: (1) Children begin at *9* and walk from there to *0*. Again, emphasize the number of steps. (2) Remove the cards, mix them, and have a child or two children put them in place. (3) Have the children close their eyes while you remove a card or two cards or interchange some. Have the children tell the names of the missing cards or reorder them properly.

Next, children should use a number line placed on the chalkboard. It can be purchased or made from a length of calculator tape. Place the line within reach of the children, so they can point to numerals *0* through *9* and indicate steps along the line with chalk. Also, a small number line should be attached to each child's desk for individual use.

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[14] Weekly Reader Family Software, 245 Long Hill Rd., Middletown, CT 06457.

[15] Random House School Division, 201 East 50th St., New York, NY 10022.

Learning the Ordinal Uses of Numbers

Number is used in the ordinal sense when the order of objects in a set is established—*first*, *second*, *third*, and so on.

Textbooks for the primary grades instruct children in the use of *first*, *second*, *third*, . . . , *middle*, and *last*. A more effective way to teach concepts involving ordinal numbers is to bring up uses of the words and their meanings whenever you can. Activities dealing with order and sequence and spatial relationships are particularly appropriate because they involve relations between and among objects. There are times during a school day when you can stress the terms and their meanings: for example, when children line up to leave or enter the room; when books are shelved by volume number; or when children learn the order of letters in the alphabet. Children often learn ordinal concepts so well through experiences like these that textbook and other written materials are unnecessary.

SPECIAL PROBLEMS WITH COUNTING AND WRITING NUMERALS

Well-equipped settings for free-play activities and well-planned developmental lessons are essential for all children. Nevertheless there are some children who do not learn to count or write numbers well and who need special attention. Some children may need an extended amount of free-play time before directed lessons begin. The purpose of each activity and the connection between one activity and another should always be made clear to help children see relationships and to organize the concepts and skills they learn. Children with handicaps may need special learning aids. Children with visual and certain physical handicaps, for example, need large-sized blocks rather than small ones. Structured rods cut from wood with a two-inch cross section can be substituted for the much smaller Cuisenaire rods. Dowel rods rather than lengths of ribbon will make it easier for some children to compare objects' lengths, while textured rather than drawn numerals will make it easier for some to learn how numerals are formed.

Certain areas of early number work are particularly troublesome to some children.

Coordination, Omit, and Double-Counting Errors

Children who commit these errors by making incorrect one-to-one matches between objects and the counting numbers should be encouraged to touch and move objects as they are counted.

ACTIVITY 6-25

Counting by Touch

Have a child count at a desk top, or on a large sheet of paper, separated into halves by a line down the middle. The child moves each object from the left side to the right side as it is counted.

This activity diminishes the chances that the child will begin or end the count inproperly, will fail to count 1 or more objects, or will double-count some. Once consistent accuracy is achieved when counting this way, have the child count objects aligned in a row, but without moving them.

Sequence Errors

Children who have difficulty remembering the sequence of numbers and the numbers that come before and after another number will benefit from *count on* activities. These activities begin at a number other than one; a child begins at a number such as "6" and counts from it.

Before count-on activities are used, each child must be aware that the last number named is the cardinal number of a set. Be sure that you and the children emphasize the last number: "One, two, *three*." Activities at a walk-on number line can stress sequence and cardinality.

ACTIVITY 6-26

Walking the Number Line II

Begin by walking the number line yourself and saying, "One, two, three, four. I took 4 steps."

Have each child walk on the line and count; then you or the child should state the number of steps the child walked.

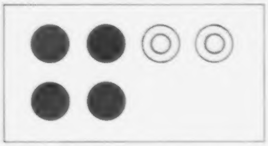
Turn over all cards following a given card, for example, 4. Have a child walk from 0 to 4; then ask, "What will be the number for the next step you take?" Turn over 5; repeat for 6, and so on.

Special practice in counting-on from small sets should be given these children.

ACTIVITY 6-27

Counting-On

Begin with a set of 4 red poker chips.



Have a child count or name the quantity on sight. After the children say "four," put down a blue chip. When the blue chip is in place, have the children say "five." Put down another blue chip and say "six." Continue until 9 chips are counted.

Variations: (1) Begin with sets other than 4. (2) Begin with a set of 5 (or some other number) and count backward as objects are removed one by one.

A teacher observed by one of the authors of this book has children pop an open palm against their forehead as "four" is spoken. This action "pops" the number into each child's head. Then as objects are added to the original set, each child holds up a finger for each number as it is counted on. To count backward, the number is again "popped" into the child's head, and a finger is lowered for each number that is counted.

Errors in Writing Numerals

Many children commit reversal errors or mistake one numeral for another while learning to write numerals. Most of them outgrow these errors; but the problems persist for some children with learning impairments, particularly those with visual handicaps, visual discrimination and spatial organization problems, and motor coordination weaknesses. Special materials for these children include large numerals cut from felt or sandpaper glued to cards and flat boxes with wet sand in which numerals are drawn. As children use these materials, there is a kinesthetic reaction that helps them remember each numeral's configuration. Writing numerals on the chalkboard and tracing them with chalk or a wet sponge are also helpful. Some children are helped by having someone trace a numeral on their back or on the back of their writing hand just before a numeral is traced or written.

NUMBER RHYMES, SONGS, AND FINGER-PLAY ACTIVITIES

Number rhymes, songs, and finger-play activities familiarize children with number names and their sequence. Children enjoy poems because of their rhythmic sound and the finger-play activities generated by many of them. Nursery-rhyme books, poetry anthologies, and teacher's manuals for some mathematics series are good sources of songs, rhymes, and poems.

Songs such as "One Little Brown Bird"[16] are enjoyed by young children. The verses of many of these songs give the numbers in sequence. The first three verses of "One Little Brown Bird" are

One little brown bird, up and up he flew,
Along came another and that made two.

Two little brown birds, sitting in a tree,
Along came another one and that made three.

Three little brown birds, then up came one more,
What's all the noise about? That made four.

Children's records are excellent sources of songs for early number work. Most children listen intently to such songs, eagerly awaiting the chance to join in when the record is repeated.

"Chickadees"[17] is an example of a finger-play.

Five little chickadees sitting on the floor;
(Hold up hand, fingers extended.)
One flew away and then there were four
(Fold down one finger as each bird flies away.)

Four little chickadees sitting in a tree;
One flew away and then there were three.

Three little chickadees looking at you;
One flew away and then there were two.

Two little chickadees sitting in the sun;
One flew away and then there was one.

One little chickadee sitting alone;
He flew away and then there was none.

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[16] *Sixty Songs for Little Children* (London: Oxford University Press, 1933). (Used by permission.)

[17] Marion F. Grayson, *Let's Do Fingerplays* (Washington, D.C.: Robert B. Luce, Inc., 1962), p. 60. (Reprinted by permission of Robert B. Luce, Inc.)

Finger-play activities can be found in some teacher's manuals, in Scott and Thompson's *Rhymes for Fingers and Flannelboards*,[18] and in Grayson's *Let's Do Fingerplays*

Summary

Mathematics begins for most children before they begin school. Their early play experiences provide a basis for later learning. They should continue to have gamelike activities in preschool, kindergarten, and the first part of grade one. Many of their activities can concern discrete and continuous materials, patterns, order and sequence, spatial relationships, and sorting and classification. Five principles govern the act of counting: the one-to-one, the stable-order, the cardinal, the abstraction, and the order-irrelevance principles. Children frequently miscount because they start or end the count improperly, they skip or double-count items, and they use idiosyncratic counting sequences.

When children are ready for counting, they must have many experiences with a variety of manipulative materials. Carefully planned activities will enable children to understand the meaning of counting and the way numerals are used to record the number of items in a collection, or set. Once children know the meaning of a number, they are ready to read and write its numeral. Reading number words should be a part of the mathematics program. There are two forms of writing numerals, the manuscript and the cursive. Manuscript numerals are used most often. Children learn to write numerals by tracing patterns or making copies alongside already written models.

Some children have difficulty learning to count and to write numerals. Special attention needs to be given to the counting process to avoid coordination, omission, and double-counting errors and to help these children learn the number sequence. Textured numerals, tracing at the chalkboard, and numerals traced on a child's back are necessary for some children as they learn to write numerals.

The computer offers opportunities for young children to interact with situations dealing with classification, spatial relationships, counting, and other mathematics concepts. The computer should not replace experiences with manipulative materials; however, it can supplement children's experiences to reinforce earlier learning and add variety to their mathematics program.

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[18] Louise B. Scott and Jesse J. Thompson, *Rhymes for Fingers and Flannelboards* (St. Louis: Webster, 1960).

Study Questions and Activities

1. Start a mathematics activities file for your teaching. An expanding file or a notebook with pockets is good because you can add ideas as you find them in your reading, classwork, or workshops. Each section or pocket may be reserved for a different type of activity. From this chapter, you may want to organize sections for counting, classifying, patterns, sets and number, and spatial relationships. The For Further Reading section also contains a list of articles suggesting activities.
2. V. Ray Kurtz (see For Further Reading section) reports that 21 percent of Kansas teachers who responded to a questionnaire about the kindergarten mathematics program indicated that the program was inadequate. Seventy-eight percent said they used an adopted series as the basis for their program, and 67 percent used a consumable workbook. Contact a kindergarten teacher and discuss the basis for this teacher's program. Is there a mandated set of objectives or an adopted program? What teaching-learning materials are used? How does the teacher feel about the adequacy of the program? What changes, if any, would the teacher make?
3. Observe or work with a group of kindergarteners who are counting small collections of objects. Notice if any of the children commit coordination, omission, or double-counting errors. Use procedures suggested in this chapter to help them overcome their errors. Which procedure(s) work(s) best?
4. Nancy Smith and Karla Wendelin (see For Further Reading section) identify 10 strands of mathematics for K–3 and list several children's books which contain mathematics concepts and vocabulary. Read 1 book each from 5 different strands, and list the terms used in each book. Describe a lesson which uses 1 of the books as the basis for a discussion with kindergarten, first-grade, or second-grade children.
5. Use the form for evaluating software in Chapter 4 as you review 2 different pieces of software designed for young children. If possible, watch young children using the software. Do the programs emphasize problem solving, active participation, and control of the computer? What concrete activities could you use to prepare the child for using the program?

For Further Reading

Carter, Beth W. "Move Over, Frank Lloyd Wright," *Arithmetic Teacher*, XXXIII, No. 1 (September 1985), pp. 8–11. Blocks provide many opportunities for mathematics thinking.

Ginsburg, Herbert P. "Children's Surprising Knowledge of Arithmetic," *Arithmetic*

Teacher, XXVIII, No. 1 (September 1980), pp. 42–44. Ginsburg cites research that shows young children exhibit surprising intellectual strengths, ranging from intuitions about *more*, to counting procedures for mental addition, to invented strategies for school arithmetic. Teachers are challenged to tap these hidden abilities.

Hamrick, Kathy B. "Are We Introducing Mathematical Symbols Too Soon?" *Arithmetic Teacher*, XXVIII, No. 3 (November 1980), pp. 14–15. Hamrick believes many children are given mathematical symbols too soon. She believes children are ready for symbols only after they can discuss a topic meaningfully.

Heddens, James W. "Bridging the Gap between the Concrete and the Abstract," *Arithmetic Teacher*, XXXIII, No. 6 (February 1986), pp. 14–17. Illustrates the use of pictures and tallies to develop numeration concepts.

Hendrickson, A. Dean. "A Psychologically Sound Primary School Mathematics Curriculum," *Arithmetic Teacher*, XXX, No. 5 (January 1983), pp. 42–47. Rationale and sample activities for the primary grades.

Hollis, Loye Y. "Mathematical Concepts of Young Children," *Arithmetic Teacher*, XXIX, No. 2 (October 1981), pp. 24–27. Summarizes research about children's knowledge of mathematics and presents a list of competencies for three-, four-, and five-year-old children.

Horak, Virginia M., and Willis J. Horak. "Developing Mathematical Understandings with Bead Strings," *Arithmetic Teacher*, XXX, No. 4 (December 1982), pp. 6–9. A set of bead strings containing 1 to 10 beads on each string provides many opportunities for children to investigate early number concepts.

Jensen, Rosalie, and David R. O'Neil. "That's Eggzactly Right!" *Arithmetic Teacher*, XXIX, No. 7 (March 1982), pp. 8–13. Plastic eggs, egg cartons, and small objects provide counting, ordering, matching, comparing, and other number experiences.

———, and Deborah C. Spector. "Geometry Links the Two Spheres," *Arithmetic Teacher*, XXXIII, No. 8 (April 1986), pp. 13–16. Movement activities for young children encourage spatial understandings.

Kurtz, V. Ray. "Kindergarten Mathematics—A Survey," *Arithmetic Teacher*, XXV, No. 8 (May 1978), pp. 51–53. Reporting on a survey made in Kansas, the author gives the competencies that are clearly questionable and those clearly not for kindergarten children. He also reports that over 75 percent of teachers use an adopted series as a basis for their program.

Lindquist, Mary M., and Marcia E. Dana. "Make Counting Really Count—Counting Projects for First and Second Grades," *Arithmetic Teacher*, XXV, No. 8 (May 1978), pp. 4–11. Counting will count for children who engage in the many counting and recording activities described as Listing Counts, Drawing Counts, Hunting Counts, Tallying Counts, Collecting Counts, and others.

Mueller, Delbert W. "Building a Scope and Sequence for Early Childhood Mathematics," *Arithmetic Teacher*, XXXIII, No. 2 (October 1985), pp. 8–11. Both number and nonnumber activities are essential in early childhood mathematics.

Shaw, Jean. "A-Plus for Counters," *Arithmetic Teacher*, XXXI, No. 1 (September 1983), pp. 10–14. Suggests many activities with counters to develop number and number system concepts.

Smith, Nancy J., and Karla Hawkins Wendelin. "Using Children's Books to Teach Mathematics Concepts," *Arithmetic Teacher*, XXIX, No. 3 (November 1981), pp. 10–15. Ten strands of mathematics concepts for kindergarten through grade three are identified, and children's books are listed for each one.

Suydam, Marilyn N. "The Process of Counting," *Arithmetic Teacher*, XXXIV, No. 5 (January 1986), p. 29. Summarizes the latest findings about how children learn to count.

Tipps, Steve, and Lynne Mann. "Computers: A New Learning Environment for Children," *Dimensions* XIII, No. 1 (October 1984), pp. 15–18. Describes considerations in the choice of software for young children and ways to prepare children for the computer with concrete materials and activities.

Van de Wall, John A. "Track Cards," *Arithmetic Teacher*, XXV, No. 6 (March 1978), pp. 22–26. Teacher-made track cards provide many kindergarten and primary-grade activities that enhance thinking, creativity, vocabulary, and understanding of topological and symmetry concepts. Includes 10 different activities with directions for preparing track cards.

Yvon, Bernard R., and Jane Dallinger Dopheide. "Iggies Come to Kindergarten," *Arithmetic Teacher*, XXXI, No. 5 (January 1984), pp. 36–38. Describes many classification activities based on a created character, Iggie.

———, and Eunice B. Spooner. "Variations in Kindergarten Mathematics and What a Teacher Can Do About It," *Arithmetic Teacher*, XXIX, No. 5 (January 1982), pp. 46–52. After surveying commercial kindergarten mathematics programs, the authors conclude that a single published program does not give children a well-rounded experience. They describe teacher-made materials for 16 activities to balance a kindergarten program.

Chapter

7

Extending Understanding of Numbers and Numeration

In Chapter 7, you will read about

1 the major characteristics of the Hindu–Arabic numeration system and their advantages for computation over other numeration systems

2 concrete materials, both nonproportional and proportional, used to develop understanding of the decimal place-value system

3 expanded notation forms for numbers that emphasize place value

4 markers and other objects that demonstrate one-to-many, many-to-one, and many-to-many correspondences used in mathematics and real-life situations

5 reasons why number theory is part of the elementary mathematics curriculum

6 activities for exploring number theory topics, such as odd-even and prime and composite numbers

7 introducing positive and negative integers on number lines

If children are to work effectively with the Hindu–Arabic numeration system, they must understand the system thoroughly and not simply have a rote knowledge of it. Therefore the topics in this chapter—the Hindu–Arabic numeration system; one-to-many, many-to-one, and many-to-many correspondences; number theory; and integers—must be presented by teachers in a manner that permits children to learn the fundamentals and rationale behind them.

THE HINDU–ARABIC NUMERATION SYSTEM

In studying our own numeration system, children do not need to be overwhelmed with historical details. Certain features of the system's history, however, are interesting to children when the topics are presented to them or when they have time to investigate them on their own or in small groups. Some teachers integrate information about the Hindu–Arabic numeration system into lessons as they present the system to children; others reserve such information as an enrichment topic for interested children.

Early development took place in India. It is likely that our present system evolved from one having 20 symbols: 9 for units, 9 for tens, 1 for hundreds, and 1 for thousands.[1] We know that numerals were used in a place-value scheme by about A.D. 600.[2] Then the forerunners of today's numerals were developed and first appeared in written records about A.D. 700. These are called Devanagari numerals (Figure 7-1).

There are conflicting statements about the date of the origin of zero. Freebury says zero appears in the Devanagari numerals of the eighth century. Smith says, "The earliest undoubted occurrence of a zero in India is seen in an inscription at Gwalior. In this inscription 50 and 270 are both written with zeros."[3] He goes on to say that the date of origin will probably never be known but that it most likely was in India in the ninth century.[4]

The Arabs' contribution to the advancement of our numeration system lies more in their transmitting information about it to other parts of the world than from their refinements of it. The Arabs translated into their language much of the knowledge of science and mathematics that had been developed and recorded in Greece, India, and elsewhere. Some of these Arabic translations are our only sources of knowledge about Greek and Indian achievements. One translation, called *The Book of al-Khowarazmi on*

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[1] Florian Cajori, *A History of Mathematics* (New York: Macmillan, 1919), p. 89.

[2] H. A. Freebury, *A History of Mathematics* (New York: Macmillan, 1961), p. 72.

[3] David Eugene Smith, *History of Mathematics, Vol. II* (Boston: Ginn & Company, 1953), p. 69.

[4] Smith, p. 69.

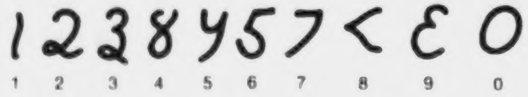

FIGURE 7-1

Devanagari numerals used in India around the eighth century

Hindu Number, explained the use of Hindu numerals. From the author's name, al-Khowarazmi, came the word *algorithm*,[5] which today means the procedures used to perform number operations. This author also wrote a book called *Al jabr*, which was about reduction and cancellation. *Al jabr* was much used in Europe, where the title was eventually corrupted into *algebra* and came to be used to describe the science of equations.[6] Today the word names one of the important branches of mathematics.

The crusades, the increased trade among nations of the Mediterranean area, and the Moorish conquest of North Africa and Spain resulted in the spread of the Hindu–Arabic numeration system to many parts of Europe. Roman numerals and the abacus, which had been spread throughout Europe during the Roman conquest, were being used there when the Hindu–Arabic system arrived. Gradually the Hindu–Arabic system and the use of algorithms were recognized, and for a time the two systems coexisted. Eventually the *algorists* won out over the *abacists*, and by the sixteenth century the Hindu–Arabic numeration system was predominant.

The characteristics of the Hindu–Arabic numeration system are summarized and explained in the following paragraphs.

1. There is a base number, which in this system is 10. During their first counting experiences, people undoubtedly used their fingers for keeping track of the count. After all 10 fingers had been used, it was necessary to use a supplementary means for keeping track of the count. It was natural that the grouping was based on ten, the number of fingers available.
2. There is a symbol for zero. It is a place holder in a numeral like 302, where it indicates that there are no tens. It is also the number that indicates the size, or numerical value, of a set that has no objects in it.
3. There are as many symbols, including zero, as the number indicating the base. The symbols in the Hindu–Arabic system are 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9.
4. The place-value scheme has a ones place on the right, a base position to the left of the ones place, a base times base (b^2) position next, a base times base times base (b^3) position next, and so on. Beginning at any place in the system, the next position to the left is ten

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[5] Freebury, *A History of Mathematics*, p. 76.

[6] Freebury, p. 77.

times greater, and the position to the right is one-tenth as large. This characteristic makes it possible to represent fractional numbers as well as whole numbers with the system.

5. The system makes it possible to make computations using only paper and pencil and a standard algorithm. Though the use of the standard algorithms will probably diminish with the advent of the calculator, their place in a mathematics program is assured for the foreseeable future because of the role they play in helping children understand the 4 operations. (Each algorithm is fully explored in later chapters.)

TEACHING THE HINDU–ARABIC SYSTEM

Classrooms at every level should be well equipped with place-value materials. Concrete manipulative materials give children the means to represent numbers greater than nine in ways that make the meanings of numbers clearer than when work begins at the pictorial or representational level of the teaching-learning sequence. A wide variety of both commercial and teacher-made materials can be used. Place-value materials are both proportional and nonproportional. Proportional materials have a physical relationship between parts that makes it possible for a child to see that 10 is ten times larger than 1, and that 100 is ten times larger than 10. There are 2 types of proportional material. The first type is material in which the student makes and breaks the grouping. A child who counts and groups tongue depressors or coffee-stirrer sticks into bundles is using type-1 proportional materials. Type-1 devices are illustrated in Figure 7-2. Unifix cubes (Figure 7-3) are colored plastic cubes with a square hole in 1 face and square nib on the opposite face. They can be joined to construct multicubed pieces of any length. They are type 1 when 10 cubes are joined to make a ten-unit piece and when 10 ten-unit pieces are put side by side to represent 100.

The second type of proportional material is one in which the grouping has already been done and the child exchanges pieces within the set. Beansticks, shown in Figure 7-4, are a teacher-made or child-made set of type-2 proportional material. A single bean represents the number one; a stick with 10 beans glued on it represents ten. One hundred is represented by 10 sticks, each with 10 beans on it, glued together to form a hundreds "raft." When beansticks are used, children exchange 10 loose beans for a tens stick as they move from single units to tens and exchange a tens stick for 10 loose beans when they trade down. The Cuisenaire rods shown in Figure 6-1 and

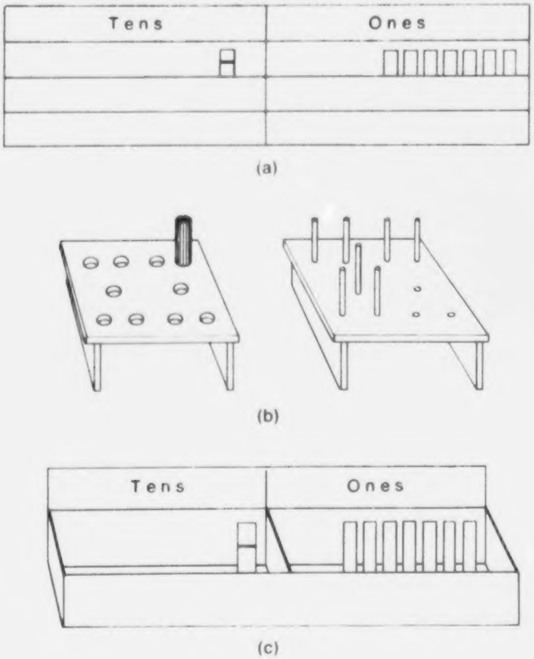


FIGURE 7-2

Place-value devices illustrating the meaning of the number 17. (a) Place-value pocket chart. (b) Place-value frames. (c) Place-value box.

the base-ten blocks in Figure 7-5 are commercial type-2 proportional materials.

Nonproportional materials do not have a proportional relationship between parts that can be visually observed by a child. Rather, the values of each piece are determined by color or position. The chip-trading materials illustrated in Figure 7-12 are nonproportional. When they are used to represent base-ten place value, 1 blue chip is equivalent to 10 yellow chips; 1

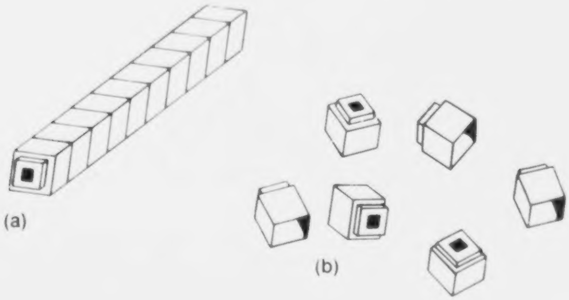


FIGURE 7-3

Unifix cubes: (a) a "train" of 10 cubes, (b) loose cubes.

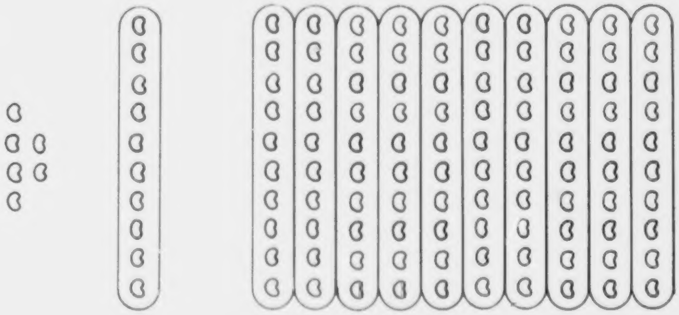


FIGURE 7-4

Set of beansticks showing (a) loose beans, (b) a tens stick, and (c) a hundreds raft

green chip is equivalent to 10 blue chips; and 1 red chip is equivalent to 10 green chips. An abacus (Figure 7-8) is also nonproportional because a single bead on a wire represents ten times the value of a bead on the wire to its right.

A progression of activities with place-value materials will occur over most of the years children are in school. Initial activities for first-grade children should involve them in work with type-1 materials. Later in the year, they can use beansticks, Cuisenaire rods, and base-ten blocks. Nonproportional materials should not be used during children's early school activities. Their use should be delayed until children have a good understanding of place value. An abacus is a useful device to help children in grade three and beyond understand larger numbers and algorithms for addition, subtraction, and multiplication.

Before you begin instruction about place value, be certain your children can do rational counting to at least 9 and can read and write the numerals for the numbers 0 through 9. They should also be able to express the numbers as addition combinations. For example, they should be able to express three as "3 + 0," "2 + 1," "1 + 2," and "0 + 3." This ability is not important for the combinations themselves but as a basis for understanding that a number like 15 can be expressed as "1 ten + 5 ones" and as "10 + 5." When children are ready for numbers greater than 9, you can use any of the type-1 proportional place-value devices. The following activity is one way to begin.

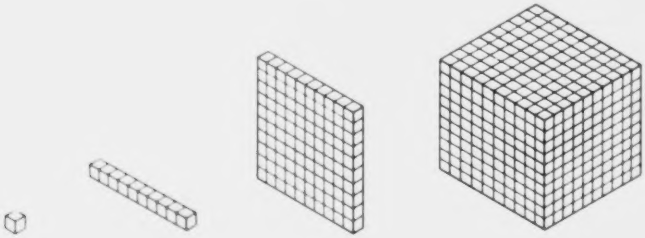


FIGURE 7-5

Pieces from a set of base-ten blocks

ACTIVITY 7-1

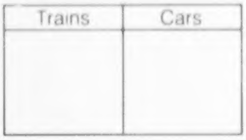
Trains and Cars

- Give each child 19 same-colored Unifix cubes.
- Tell the children that they are to join cars (single cubes) to make trains (a 10-unit piece). When a train has been made, have children count a given number of the remaining cubes—anywhere from 1 to all 9. Each time a train and loose cars are made, ask questions so that the children must think about the results of their work: "How many trains do you have?" "How many cars do you have?" "How many trains and cars do you have?"

Children will indicate through words and actions when their understanding of a concept is becoming internalized. Their descriptions of activities will become fluent and their actions with materials sure. When their actions with Unifix cubes and descriptions of what they are doing indicate that they understand the underlying concept of grouping by 10, you can introduce a train/car mat on which they can perform their actions (Figure 7-6). Now each child can make a set of trains and cars and place them on the mat to indicate their values. Later they can record the count with numerals, either by placing cards containing numerals at the bottom of the mat, or by writing the numerals on a similar mat duplicated on paper.

FIGURE 7-6

Mat for keeping track of trains and cars



Activities 7-2 and 7-3 extend children's place-value activities to introduce numerals for identifying numbers greater than 10. Activity 7-2 is a counting activity, in which children are introduced to the numerals 10–19. In Activity 7-3, they use beansticks to further their understanding of place value.

ACTIVITY 7-2

Place Value with Sticks and Chips

- Use a place-value device with sticks, such as one of those in Figure 7-2, and chips on a magnetic board. Put 15 chips on the board.

- Select 3 children: The first to count the chips, the second to represent the count on the device, and the third to record the count with numerals on a chalkboard.
- As the first child counts the chips one by one, the second child puts a stick for each chip in the ones place of the device. At the same time, the third child records the count with numerals written in sequence on the chalkboard.
- Pause at nine and review what has been done so far.
- As the tenth chip is counted and the tenth stick is placed in the ones place, ask, "How many chips has Sue counted?" "How many sticks are in the ones place?" "How can Bob write '10' on the chalkboard?" Demonstrate how to bundle 10 sticks to represent 1 ten, and place the bundled sticks in the tens place. "How many tens?" "How many ones?" Explain how "10" is written and its meaning.
- Continue the count by representing "11" with a bundle of 10 and a single stick and showing how "11" is written. Do the same with 12, 13, 14, and 15.
- Discuss the meaning of "11" as 10 and 1, "12" as 10 and 2, and so on.
- Repeat with different numbers and other children as counters and recorder.

ACTIVITY 7-3 Place Value with Beans and Sticks

- Give each child 19 loose beans, a tens beanstick, and a place-value mat.

Tens	Ones

- Tell each child to count 13 loose beans and put them in the ones place on the mat; talk about the number of beans and point out that there are more than 9.
- Tell each child to separate the beans into a group of 10 and a group of 3, and to put the 10 beans in the tens place of the mat.
- Have each child exchange the 10 loose beans for one tens stick; discuss the fact that there are 1 ten and 3 ones, or 13.
- Write the expanded notations 1 ten and 3 ones = 13 and 10 + 3 = 13 on the chalkboard and discuss.
- Repeat with other numbers between 9 and 20.

Frequent activities, both teacher-led and individual and small group work spread over several months, are necessary for most children. Children who demonstrate skill and understanding while working under your direction are ready for individual and cooperative work with work sheets, problem cards, and games. Work sheets from children's workbooks can be used for individualized practice. Some work sheets contain pictures of common place-value devices; later the work becomes more abstract. You must be certain that your children have both pedagogical and content readiness before they do workbook exercises. Problem cards like those in Figure 7-7 are particularly good because they can be adapted to match the concrete place-value devices that children used during initial instruction. Naturally you will want more than 1 card for each material. Teacher guides for commercial materials contain many activities dealing with place value.[7]

After children understand place value through 19, they should continue to work with manipulative materials that show place value through 99. Activities for these numbers are the same as for numbers through 19. Once understanding has been developed, a game offers the opportunity for children to practice a particular skill in a group situation. In every grade a good game contributes to learning by giving children additional manipulative experiences with familiar materials. The changed setting in which the materials are used enlivens the practice session and heightens children's interest in maintaining their skills. An example of a place-game for numbers between 9 and 100 is Beans and Sticks.

GAME 7-1

Beans and Sticks

Materials: A set of 18 loose beans, 9 tens sticks, and a place-value mat for each player; one 0–9 spinner.

Players: 2–4

Order of Play: Each child spins the spinner; use either high or low to determine the first player.

Rules: The first player spins the pointer and calls out the number at which it stops. The player counts as many beans and puts them in the ones place of the mat. Each player takes a turn. The first player spins again, counts beans, and puts them on the mat. If the count exceeds 9, the player exchanges 10 loose beans for 1 tens stick and places it on the mat in the tens place. At the end of each play, the child tells the number represented on her or his mat. Play continues until one player has a score in the 90s, or until an agreed-upon number of spins have been made.

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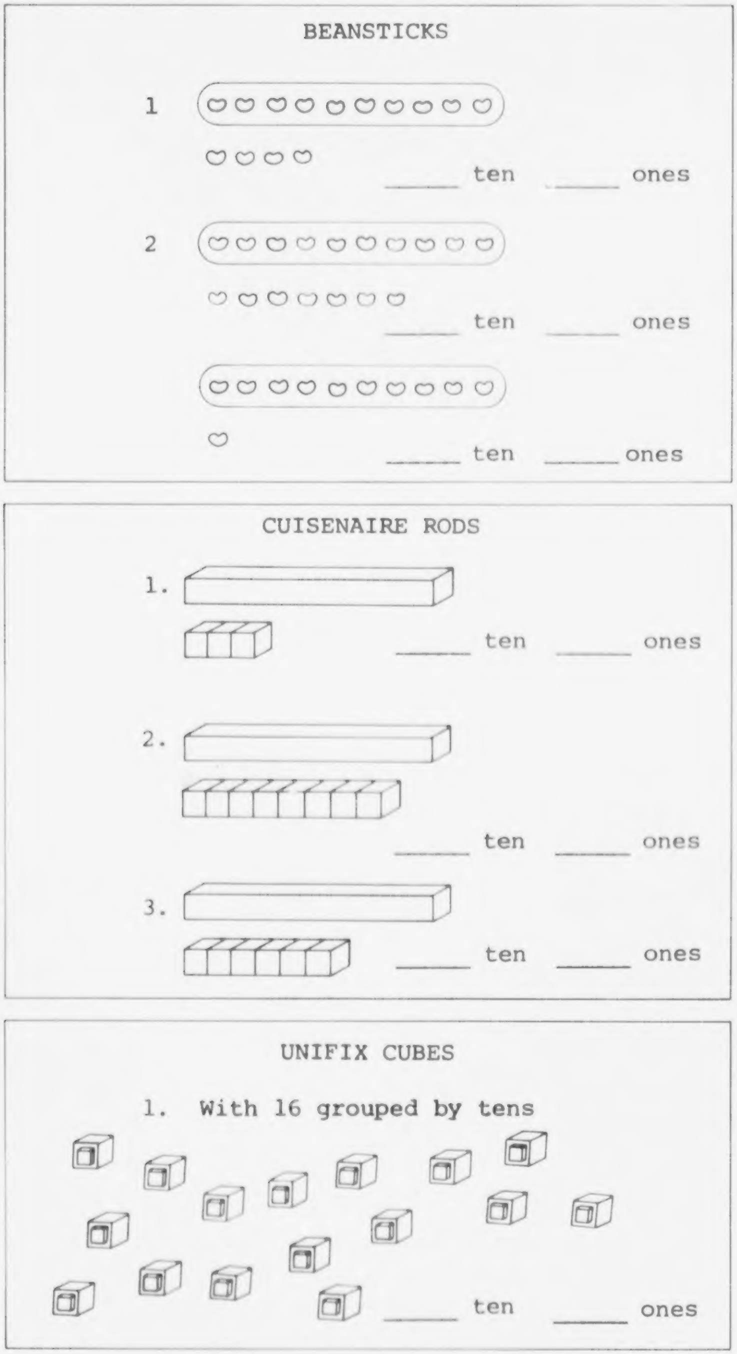
[7] See *Using the Cuisenaire® Rods—A Photo/Text Guide for Teachers*, *Hands on Base Ten Blocks*, Books 1, 2, and 3; and *Hands on Unifix Cubes*, Books 1, 2, and 3. Creative Publications. 5005 West 110th Street, Oak Lawn, IL 60453.

FIGURE 7-7

Problem cards for activities with structured materials


BEANSTICKS
1
_____ ten _____ ones
2
_____ ten _____ ones
_____ ten _____ ones
CUISENAIRE RODS
1.
_____ ten _____ ones
2.
_____ ten _____ ones
3.
_____ ten _____ ones
UNIFIX CUBES
1. With 16 grouped by tens
_____ ten _____ ones

Variation: Use hundreds rafts and 2 spinners, 1 designated tens and 1 designated ones, with mats showing ones, tens, and hundreds.

In later grades, children work with larger numbers. Their understanding of the meaning of numbers through 999 should be developed as they engage in activities similar to those used in learning about the numbers through 99. It is not necessary to continue counting objects, but the place-value devices will remain useful in illustrating that 10 tens equal 1 hundred as 10 bundles of 10 markers each are put together to make 1 bundle of 100 in one of the devices.

The flats and rods in Cuisenaire and base-ten sets, and rafts and tens sticks in a set of beansticks, are used to represent numbers between 99 and 999. Cubes are used with flats and rods for numbers between 999 and 9999. Special plastic grids are used to make squares and large cubes of Unifix cubes.

As children work with a device, they also learn how the numeral for 10 tens, or 100, is written. They should note the relationship between the symbol "100" and the markers on the place-value device, a single flat in a set of structured material, or a beanstick raft. As children continue this study, special attention should be given to the tens place when it contains a zero, as in 101. Some of the larger numbers should be practiced until each student understands the place-value concept.

Activity 7-4 provides an interesting setting in which to count a large number of objects and to give children experience in making estimates.

ACTIVITY 7-4

Pumpkin Seeds

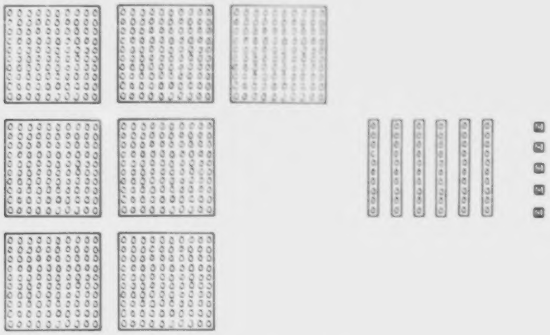
Materials: One medium-sized pumpkin, dark-colored art paper, glue.

Display the pumpkin for a few days to allow each child time to estimate the number of seeds it contains. Have each child record a guess and give it to you.

Open the pumpkin and remove the seeds. Separate the seeds from the pulp and dry them.

Use a place-value scheme to record the count. One scheme is to have children count sets of 10 seeds until 100 have been counted. (This provides practice in counting by ones and tens.) Glue each collection of 100 seeds on a square piece of the art paper. If there are more than 1,000 seeds, put 10 cards of 100 beans on a larger piece of paper to show 1,000. When fewer than 100 seeds remain to be counted, group them by tens until there are fewer than 10. Glue each set of 10 seeds on strips of paper. Finally, glue the single seeds on small squares of paper.

Arrange the cards with the hundreds (or thousand) on the left, then tens and ones. Have the children determine the number of seeds. Write the number with expanded notation with words and with numerals, and in compact form. Example:



7 hundreds + 6 tens + 5 ones

700 + 60 + 5

765

Check the estimates to determine which was the nearest to the correct number. Discuss the fact that some estimates were close to being correct and (if there were some very small or very large numbers) that some were too far on either side to be reasonable.

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*Sum Ducks** is a computer program for addition and subtraction of whole numbers, but the format of the game emphasizes grouping by tens. The ducks in the pond are worth either 1 or 10 points. For each game, the computer displays a random target number. The student uses a joystick to throw rings at the tens and ones ducks in an attempt to accumulate a sum equal to the target number.

A classroom abacus is an excellent device for extending children's understanding of place value to larger numbers, because it can show numbers that are impractical with other devices. To make one for classroom use, insert pieces of stiff wire into holes drilled in a board; wooden beads can be used for markers. Commercially made abacuses are also available. The abacus in Figure 7-8 has enough rods to represent numbers as large as billions.

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* Spinnaker Software, 215 First S et, Cambridge, MA 02142.

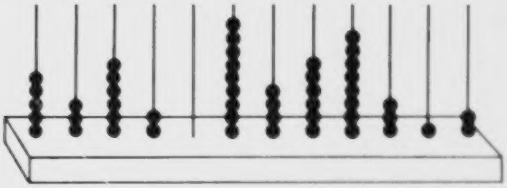


FIGURE 7-8

Abacus representing the number 536,209,468,312

A first abacus should have only 3 or 4 rods. You can use this abacus to demonstrate to third or fourth graders how it represents numbers. Point out that 1 bead on the tens rod has the same value (that is, represents the same number) as 10 beads on the ones rod (10); 1 bead on the hundreds rod represents the same number as 10 beads on the tens rod (100). Under your guidance, children will conclude that each rod represents a place-value position that has a value ten times greater than the position to its immediate right, regardless of where it is on the abacus. By the time they are in the fourth grade, most children will, with your help, be able to reverse their thinking to conclude that each place-value position also represents a value that is one-tenth the value of 1 to its immediate left. As their understanding of place value increases, children will generalize that the Hindu–Arabic numeration system has a place-value scheme based on 10 and powers of 10. This same scheme forms the basis of the metric system, a fact that should be capitalized upon as children deal with metric measure.

A place-value chart is a useful means for helping intermediate-grade children understand large numbers (Figure 7-9). Numerals instead of beads are used to represent each place-value position. Children should read the numbers represented on the chart and also put numerals on the chart to represent numbers presented orally.

A calculator is useful for helping children learn about place value for large numbers. As children work with abacuses and place-value charts, you can have them enter into a calculator the numbers with which they are working. Show an abacus like the one in Figure 7-8 and ask children to enter the number represented on it into their calculators. Remember that numbers greater than 99,999,999 cannot be displayed on children's calculators. You

FIGURE 7-9

Place-value chart showing value of place positions for the number 536,209,468,312

Hundred billions	Ten billions	Billions	Hundred millions	Ten millions	Millions	Hundred thousands	Ten thousands	Thousands	Hundreds	Tens	Ones
5	3	6	2	0	9	4	6	8	3	1	2

may need to demonstrate how to enter large numbers the first few times children do this. Once children learn to enter numbers, the "wipe-out" activity that follows can be used to check their understanding of place value in large numbers. Third- and fourth-grade children can use numbers like those in Activity 7-5. Older children can use numbers into the millions, where they wipe out numbers from any place-value position.

ACTIVITY 7-5

Wipe Out

- Children work singly or in pairs with calculators.
- The object of this activity is to "wipe out" a number in a particular place-value position in 1 move. For example, when shown the number 547, the child is to wipe out the 4 tens with one subtraction. In order to do this, the child must recognize that the "4" represents 4 tens, or 40; 40 is subtracted from 547, leaving 507.
- Write numbers like these on a chalkboard, or present them orally:

Wipe out tens	Wipe out hundreds	Wipe out thousands
798	2379	47,956
321	9999	71,233
475	7524	63,129

- Variation: Name a number such as seven thousand for children to display on their calculators. Call out values to add, such as "Add 4 tens," "Add 8 ones," and "Add 3 hundreds." Then have children compare their totals.

A game that alerts students to place value can be played with dice or spinner. Students must develop a strategy for putting numerals in place-value positions to make the largest number.

GAME 7-2

Spin to Win

- Materials: a spinner with the numerals 0–9. (Spinners are available from school supply companies.)
- Have each student draw four lines (_______ _______ _______ _______) on a piece of paper to represent a 4-digit numeral.
- Turn the spinner 4 times to get 4 numbers to fill in the blanks. Each number must be used as it occurs; children cannot accumulate numbers and place them on lines after all four have been identified.
- Challenge students to fill the blanks with the 4 numerals to get the largest possible number.

- All students who record the largest possible number must read the number correctly to earn a point.
- Variations: (1) Make the smallest possible number with 4 spins. (2) Expand the number of blanks to 5 or more place-value positions.

EXPRESSING NUMBERS WITH EXPANDED NOTATION

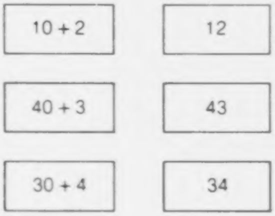
Children use both compact and expanded forms of numerals in elementary-school mathematics. In the compact form, the one we normally use, the number 243 is written as "243." There are times, however, when one of the expanded forms is useful. For example, when children are learning about place value, the expanded forms *2 hundreds + 4 tens + 3 ones* and "200 + 40 + 3" emphasize place value in the Hindu–Arabic numeration system. Also, it is best that primary-grade children know how to express numbers in the simplest expanded form before beginning to add and subtract numbers represented by multidigit numerals, because this form helps children understand the algorithms better. Intermediate-grade children need to understand expanded notation, particularly in its exponential form, because it serves as a foundation for the study of extremely large and small numbers and their expression in scientific notation.

The simplest form of expanded notation is the expression of a number using a combination of numerals and words; for example, the number 17 is expressed as "1 ten and 7 ones." As children count objects and use place-value devices, you should repeatedly have them interpret the numbers as so many tens and so many ones, and write the numerals in both the standard and simple expanded forms. From the use of the form "5 tens and 6 ones" to express the number 56, children move to the shorter "50 + 6" form. A game such as Expanded Notation Concentration provides practice in recognizing and naming expanded forms of numbers.

GAME 7-3

Expanded Notation Concentration

Materials: 8 to 12 pairs of cards like those shown.



- Players: 2–3
- Order of Play: Players determine order before the game begins.
- Rules: Shuffle the cards and put them face down on a desk in rows and columns to form a rectangle. The first player turns over a pair of cards and determines if they match. If they do, the player says, "10 plus 2 equals 12," or whatever the numbers are. This pair is kept by the player, who takes another turn. If a pair does not match, the cards are returned to their places on the desk, and the next player takes a turn. Play continues until all pairs have been matched. The player with the most pairs is the winner.

In the second through fourth grades, children will apply the expanded form to larger numbers, such as 53,692. As they use an abacus to represent such a number, fourth-grade children should learn to express it in the following ways:

5 ten thousands, 3 thousands, 6 hundreds, 9 tens, and 2 ones
53 thousands, 6 hundreds, 9 tens, and 2 ones

$$50{,}000 + 3{,}000 + 600 + 90 + 2, \text{ or } \begin{array}{r} 50{,}000 \\ 3{,}000 \\ 600 \\ 90 \\ \underline{2} \\ 53{,}692 \end{array}$$

After children can multiply by ten and powers of ten, they are ready to express numbers in two other forms:

$$(5 \times 10{,}000) + (3 \times 1{,}000) + (6 \times 100) + (9 \times 10) + (2 \times 1)$$

$$[5 \times (10 \times 10 \times 10 \times 10)] + [3 \times (10 \times 10 \times 10)] \\ + [6 \times (10 \times 10)] + [9 \times 10] + [2 \times 1]$$

When children understand the meanings of these forms, they will have little difficulty learning how to express numbers by exponential notation. It is usually in the sixth grade that children learn to use exponents to indicate powers of 10. In exponential notation, 53,692 is expressed in this way:

$$(5 \times 10^4) + (3 \times 10^3) + (6 \times 10^2) + (9 \times 10^1) + (2 \times 10^0)$$

REGROUPING

$$\begin{array}{r} 46 \\ -19 \\ \hline \end{array}$$

$$\begin{array}{r} \overset{3}{\cancel{4}}\overset{1}{6} \\ -19 \\ \hline \end{array}$$

Regrouping is the process used to express a number in a form that is convenient for computation. For example, when the algorithm in the margin is computed using the decomposition algorithm, it is necessary to regroup 46 as 3 tens and 16 ones so that 9 can be subtracted. The regrouping is shown by marking out the 4, recording a 3 in its place, and showing 16 ones. You should help children understand the regrouping process through activities with place-value devices before the process is used to compute algorithms. When beansticks are used, have children make exchanges both ways. That is, in addition to grouping loose beans to exchange for tens sticks, have them exchange tens sticks for loose beans. For example, show a set of 4 tens sticks and 6 loose beans. Demonstrate how 1 tens stick can be exchanged for 10 loose beans, giving 3 tens sticks and 16 loose beans. The first variation of the Banker's Game (Game 7-4) provides opportunities for children to regroup tens as ones as they trade down. The Beans and Sticks Game (Game 7-1) can also be played as a "countdown" game.

An abacus is useful for showing regrouping with numbers greater than 999. For this purpose, an abacus must have room for 18 beads on a wire. A commercial abacus that has this capacity is available (Figure 7-10). The teacher-made abacus in Figure 7-11 shows 23,498. It appears in its simplest

FIGURE 7-10

Commercial abacus showing the number 2,579

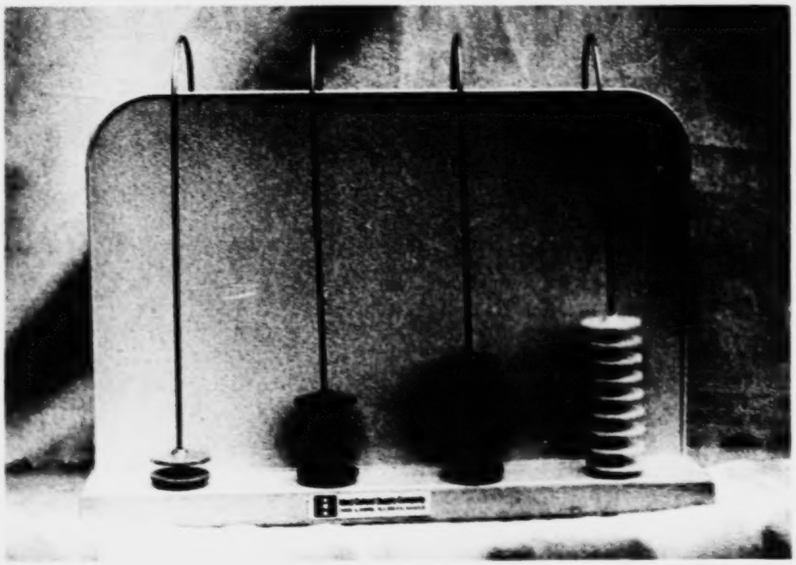
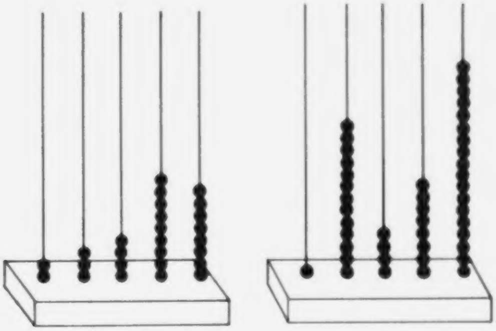


FIGURE 7-11

Abacus representing 23,498—in (a) as 20,000 + 3000 + 400 + 90 + 8; in (b) as 10,000 + 13,000 + 400 + 80 + 18



(a) (b)

form in (a) and in one of its many regrouped forms in (b). The second representation is useful either to show the sum of the addition problem

$$\begin{array}{r} 17{,}349 \\ \underline{+\ \ 6{,}149} \end{array}$$

before it is regrouped or to prepare for subtraction using the decomposition (borrowing) method of the problem:

$$\begin{array}{r} 23{,}498 \\ \underline{-\ \ 9{,}129} \end{array}$$

Particular attention to numbers with zeros builds readiness for regrouping when such numbers are used in subtraction. If children learn that 204 can be expressed as "20 tens and 4 ones," 3006 as "300 tens and 6 ones," and 40,002 as "4000 tens and 2 ones," they will not be confused by this regrouping:

$$\begin{array}{rcrcr} 3006 & \longrightarrow & 300 \text{ tens and } 6 \text{ ones} & \longrightarrow & 299 \text{ tens and } 16 \text{ ones} \\ \underline{-2439} & \longrightarrow & \underline{-(243 \text{ tens and } 9 \text{ ones})} & \longrightarrow & \underline{-(243 \text{ tens and } \ \ 9 \text{ ones})} \end{array}$$

The subtraction can begin after one regrouping. This procedure is easier to do and understand than to regroup step by step from the thousands place to the hundreds place, from the hundreds place to the tens place, and finally from the tens place to the ones place. Such a lengthy process is unnecessary for children who can represent numbers in many expanded forms.

SPECIAL PROBLEMS WITH NUMERATION

Some children have problems learning to count objects in sets greater than 10 and writing numbers larger than 10. For example, a child who has perceptual problems may have difficulty keeping his or her place while counting or may reverse digits in a number. If the child's deficit is auditory, numbers like 14 and 40 may not be easily distinguished. Your awareness of a child's handicap will give you the knowledge you need to adjust instruction to accommodate the student's deficiency. One way to help students who have difficulty keeping the place while counting is to use the following activity:

ACTIVITY 7-6

Counting by Using a Hundreds Chart

Count by ten using a hundreds chart. As the children count, emphasize the pronunciation of each number beyond ten: twenty, thirty, forty, and so on. Write the numerals in sequence: 10, 20, 30, 40, 50, 60, 70, 80, 90.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Locate each number with 9 in the ones place and ask, "What number comes after 29?" "What number follows 49?" Help the children note that in each case, the tens number increases by 1.

Give each child 40 or 50 lima beans to count. Observe each child to note if the transition is made properly.

Variation: Use a number line instead of a hundreds chart.

Many children reverse digits when writing two-place numbers; the teens are particularly troublesome for many. When these writing errors persist, special attention should be focused on the number in the tens place.

Nancy S. Bley and Carol A. Thornton recommend that color cues be used. After a number like 23 has been represented with sticks or some other device, write the number with a green "2" and a red "3." Tell the children to think of a stoplight, in which green means *go* and red means *stop*. Green and red chips can also be used to represent numbers, with green for tens and red for ones. When children write numbers, have them use a green crayon for tens numbers and a red crayon for ones numbers. Color cues should be used until children are able to write numbers consistently and correctly.[9]

ONE-TO-MANY, MANY-TO-ONE, AND MANY-TO-MANY CORRESPONDENCES

Counting is a process of making a one-to-one correspondence between a collection of objects and the natural numbers 1, 2, 3, ... It is important that children understand one-to-one correspondence as the basis for counting. It is important that they recognize that one-to-many, many-to-one, and many-to-many correspondences have useful applications too.

Examples of one-to-many correspondences are place value (1 ten is equal to 10 ones), money (1 nickel is equivalent to 5 pennies), linear measure (1 meter is equivalent to 100 centimeters), and weight measure (1 gram is equivalent to 1000 milligrams). Many-to-one correspondences are also illustrated by place value (10 tens equal 1 hundred), money (2 nickels are equivalent to 1 dime), linear measure (12 inches are equivalent to 1 foot), and weight measures (16 ounces are equivalent to 1 pound).

Many rate-type problems are based on these same correspondences. An example is a problem involving distance traveled. "A bicyclist can travel at an average rate of 12 miles in 1 hour. What is the time required to travel 36 miles?" The rate of travel is a many-to-one correspondence, and the answer is a many-to-many correspondence. The rate in the problem "If 2 pencils cost 15 cents, how many can you buy with 60 cents?" is a many-to-many correspondence, as is the answer.

Commercial and teacher-made activities will help children understand these correspondences. *Chip Trading Activities* kits[10] offer a variety of activities.

A kit contains 4 colors of circular chips, chip boards, chip tills, abacus boards, dice, numeral and operations cards, and manuals (Figure 7-12). The

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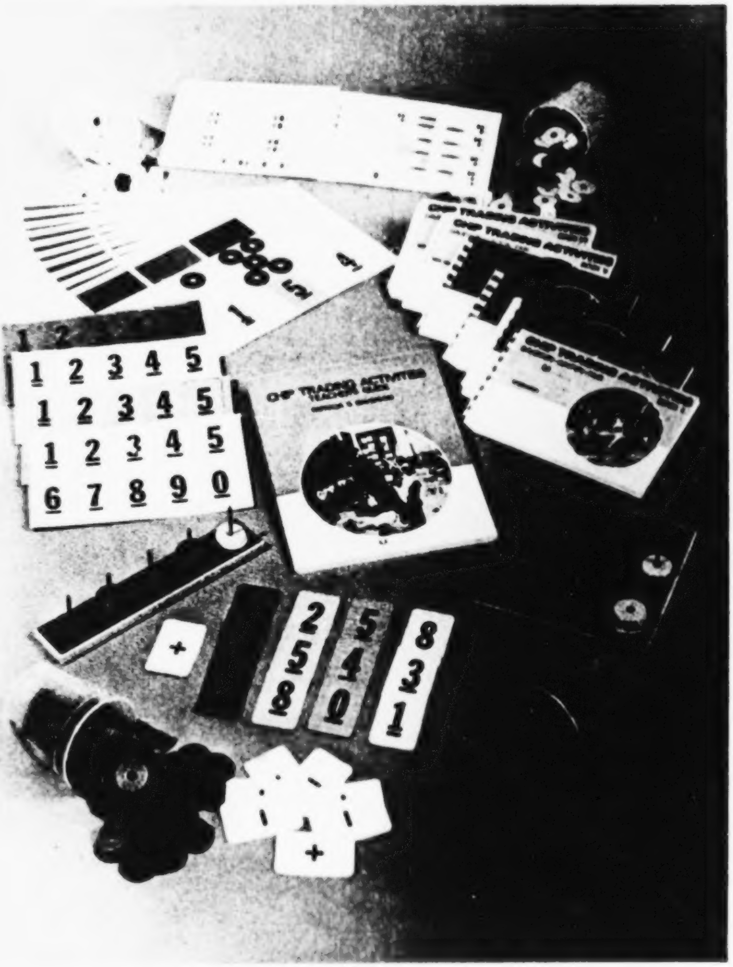
[9] Nancy S. Bley and Carol A. Thornton, *Teaching Mathematics to the Learning Disabled* (Rockville, Md.: Aspen Systems Corporation, 1981), pp. 56–65.

[10] Patricia Davidson, Grace Galton, and Arlene Fair, *Chip Trading Activities* (Fort Collins, Colo.: Scott Resources). Used with permission.

FIGURE 7-12

Chip-trading materials. Photograph courtesy of Scott Resources, Fort Collins, Colorado.


1 2 3 4 5
1 2 3 4 5
1 2 3 4 5
6 7 8 9 0
CHIP TRADING ACTIVITIES
2 5 8
5 4 0
8 3 1
+

Banker's Game provides opportunities for children to practice many-to-one and one-to-many trades.[11]

GAME 7-4

Banker's Game

Materials: 1 red, 15 each green, blue, and yellow chips, 1 chip board for banker, 1 chip till for each player, 1 die.

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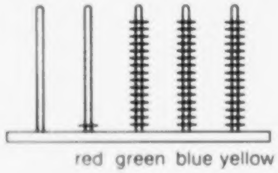
[11] This game is adapted from Davidson, Galton, and Fair, *Chip Trading Activities, Book I*. Scott Resources. Used with permission.

Players: 3–5 (one acts as banker).

Order of Play: Roll the die, with highest or lowest roll going first.

Rules: Banker sets up bank as shown.



Chips are traded this way:

3 yellow = 1 blue
3 blue = 1 green
3 green = 1 red

The first player rolls the die and asks the banker for the number of chips indicated on it. If the number is 5, the player gets 5 yellow chips. Three of the yellow chips are exchanged for 1 blue chip; all chips are placed on the chip till. Next players have their turns in order. Play continues until 1 player gets a red chip.

R	G	B	Y

Variations: (1) Play can begin with each player having 1 red chip. Players trade down until 1 player clears her or his till. (2) The rate of exchange can be different. A ten-for-one rate gives practice with the base-ten scheme.

Yellow, blue, green, and red spray-painted lima beans are an inexpensive substitute for chip-trading materials. A chip till can be outlined on a piece of manila tagboard.

SKIP-COUNTING

Skip-counting is used to count objects by a sequence other than ones, such as counting by twos, fives, and tens. It is a useful way to count money, vote tallies, and large collections of objects. Skip-counting is also used when multiplication is interpreted as repeated addition. Children's first work with skip-counting should build on their knowledge of patterns.

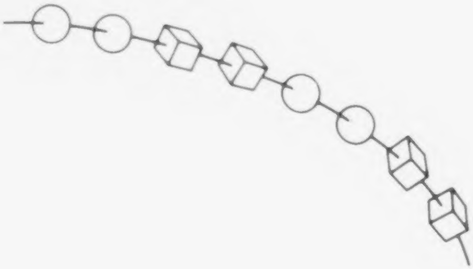
ACTIVITY 7-7

Skip-Counting

Show children a bead or shape pattern like the ones in Figure 6-2; discuss each pattern.

Show a pattern that alternates by twos, and have children note the pairs of objects. Count by ones.

Ask, "Does anyone know another way to count the beads?" If counting by twos is not mentioned, show how it is done.



Count other pairs, such as pennies grouped by twos, pairs of desks placed side by side, and pairs of children in a line.

Do the same with patterns of 5 and 10.

ACTIVITY 7-8

Patterns

Study patterns and skip-counting on a hundreds board.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Examine rows across to observe how counting by twos includes every other space, counting by fives includes every fifth space, and counting by tens includes every tenth space.

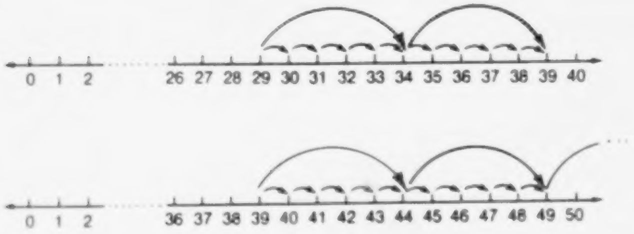
Look at the columns. The number in the ones place is the same in each column, whereas the number in the tens place increases by 1.

A diagonal from top left to bottom right increases by 11, and one from top right to bottom left increases by 9. Study these diagonals to determine why these patterns occur.

Skip-counting can also be done on a number line. A line on calculator tape that extends into the hundreds can be placed above chalkboards and bulletin boards on one or more classroom walls. Mark numbers ending in 5 with red ink and those ending in 0 with blue ink to make it easy to count into the hundreds by fives and tens. Also, have children skip-count by a number beginning at a number that is not a multiple of the counting number. Figure 7-13 shows a part of a count by 5 beginning at 29.

FIGURE 7-13

Number line used to illustrate skip-counting by fives from 29



Patterned skip-counting is beneficial and fun. Rather than using a constant skip count, use one like 1, 5, 7, 11, 13, 17, ...

Skip-counting is also possible with a calculator.

ACTIVITY 7-9

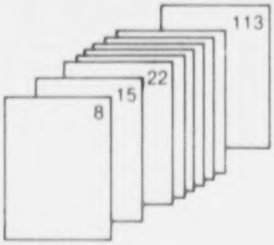
Calculator Counting

Children work singly or in pairs with a calculator.

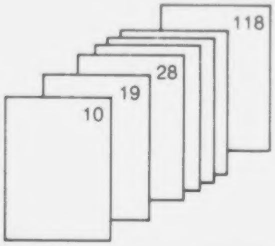
Tell the children that they are to figure out how to make the calculator count by ones. When the children have determined how to do this, have them count by twos, fives, and tens.

Challenges: (1) "Can you count by fives to 500 in a minute?" (Have 1 child time another with the second hand of the classroom clock or with a digital timer.) (2) Give children problem cards with activities like the following:[12]

Find how many cards are in this stack. Use the code 1 + 7 = = =, . . .



How many cards are in this stack? Try this code: 1 + 9 = = =, . . .



ELEMENTARY NUMBER THEORY

According to Dantzig, number theory was developed before the theory of arithmetic.[13] Long before efficient ways to use algorithms were worked out, people began to study the relationships among numbers. Number lore, the study of prime and composite numbers, and other aspects of number theory have a long history. Even so, ". . . the theory of number is the branch of mathematics which has found the least number of applications. Not only has it so far remained without influence on technical progress, but even in the domain of pure mathematics it has always occupied an isolated position only loosely connected with the general body of the science."[14]

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[12] Adapted from Project Impact, *Problem Solving Using the Calculator, Book I* (Cedar Falls, Iowa: Price Laboratory School, University of Northern Iowa, 1982), p. 3.

[13] Tobias Dantzig, *Number, the Language of Science*, 4th ed. (New York: Doubleday, 1954), p. 38.

Today number theory has achieved a new position in the elementary-school program. Where formerly the study of number theory was incidental and haphazard, if it existed at all, now it is planned and organized to lead children to discover certain elementary generalizations about numbers. There are at least 3 reasons for its present position:

1. The study helps children understand that number theory can be useful in other areas of mathematics. Children who know about prime and composite numbers can use them to factor composite numbers and find the *greatest common factor* (GCF) and the *least common multiple* (LCM) of 2 or more numbers. Later they can use this knowledge to find common denominators and to simplify, or reduce to *lowest terms*, common fractions that represent fractional numbers.
2. Many children find that work with number theory is interesting and fun. Some find that work with prime and composite numbers leads to many interesting areas of investigation.
3. During the study of different aspects of number theory, many opportunities arise to work with the basic facts of the 4 operations, to practice computational skills, and to further develop a mathematics vocabulary. For example, the process of factoring numbers involves factors, or divisors, and multiples of numbers.

Many topics dealing with number theory are well suited for learning centers, with directions for activities on problem cards or tapes for playback on cassette recorders. Activities and materials, some in a problem-card format, are suggested here.

1 Odd and Even Numbers

Investigations that deal with whole numbers to give children an understanding of odd and even numbers can begin in the primary grades. Children can use disks, cubes, or other objects as manipulative materials to complete an activity like the following.

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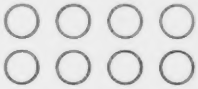
[14] Dantzig, p. 38.

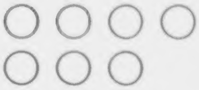
ACTIVITY 7-10

Even and Odd

- Have children work in pairs, using disks.
- Tell them to arrange 8 disks in 2 rows, with the same number in each row. Count the number in each row.



- Repeat with 10, 2, 6, 14, 30, 18, and other even numbers.
- Have children make other sets that have 2 rows with an equivalent number in each row.
- Have children arrange 7 disks in 2 rows so that 1 row contains an extra disk. Emphasize that the rows do not contain the same number of disks.



- Repeat with 5, 11, 15, 21, 1, and other odd numbers. Emphasize that the rows never contain the same number of disks.
- Have children make other sets that cannot be put in rows with the same number of disks in each row.
- Have children identify the numbers that formed 2 equivalent sets (2, 4, 6, . . .) and those that have an extra disk in one row (1, 3, 5, . . .). Tell them that the numbers 2, 4, 6, . . . are even numbers, and the numbers 1, 3, 5, . . . are odd numbers.

Follow this activity with investigations on the number line. Have children locate odd and even numbers on the line and then count by twos beginning at an even or an odd number and going forward or backward. The forward count can go to the end of the line or to a designated number such as 14 or 19. The backward count can go to 0 or 1 or to any designated number such as 3 or 6.

A hundreds chart offers many opportunities for children to extend their experiences with odd and even numbers. (A hundreds chart is illustrated in Activity 7-12.) Counting activities similar to those done with the number line can be done with the chart. These observations are among the ones children will make:

Beginning at 2, every other number is an even number; beginning at 1, every other number is odd.

The column on the left side of the chart contains all odd numbers ending in 1. Each alternate column across the chart contains odd numbers ending in 3, 5, 7, and 9.

The second column from the left contains all even numbers ending in 2. Each alternate column across the chart contains even numbers ending in 4, 6, 8, and 0.

The even numbers are multiples of 2.

Intermediate-grade children can study odd and even numbers at a more abstract level. These generalizations can arise from their investigations:

If an even number is designated by $2n$, then its successor is $2n + 1$ and is an odd number. ($2n$ is used to designate an even number, because every even number has 2 as a factor; therefore 2 times any number must be even.)

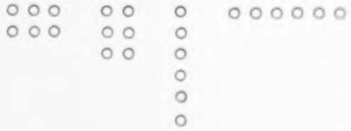
Zero is an even number.

Prime and Composite Numbers

Intermediate-grade children can investigate the nature of prime and composite numbers. Activities with poker chips or other disks allow children to investigate different array patterns. The following activities can be presented in a series of teacher-directed lessons.

ACTIVITY 7-11 Prime and Composite Numbers

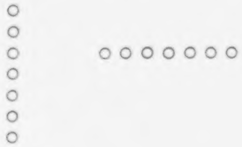
Arrange sets of 4, 6, 8, 9, 10, and 15 disks into all of their possible arrays. (The arrays for 6 disks are shown.)



Keep a record of the arrays for each set of disks.

Whole Number	Arrays	Number of Different Arrays
6	2 by 3, 3 by 2, 6 by 1, 1 by 6	4

- Make arrays for sets of 2, 3, 5, 7, 11, 13, and 17 chips and record the results. (The arrays for 7 disks are shown.)



- Determine the array for 1 disk and record it.
- Have the children examine the chart for generalizations about different arrays and factors for the numbers. The following chart helps children make these generalizations:

Whole Number	Arrays	Number of Different Arrays
1	1 by 1	1
2	1 by 2, 2 by 1	2
3	1 by 3, 3 by 1	2
4	2 by 2, 1 by 4, 4 by 1	3
5	1 by 5, 5 by 1	2
6	2 by 3, 3 by 2, 6 by 1, 1 by 6	4
7	1 by 7, 7 by 1	2
8	2 by 4, 4 by 2, 8 by 1, 1 by 8	4
9	3 by 3, 9 by 1, 1 by 9	3
10	2 by 5, 5 by 2, 10 by 1, 1 by 10	4

- Some whole numbers—2, 3, 5, and 7—have only 2 arrays; these form a straight line rather than a rectangular pattern. The numerical description always contains a 1 and the numeral that tells the total number of markers in the array.
- Some whole numbers—4, 6, 8, 9, and 10—have more than 2 arrays. They can be arranged in 1 or more rectangular patterns (square in the

case of 4 and 9). Numerals other than 1 and the numeral that names the number of markers in the set are used in the numerical expressions for these arrays.

☐ Only 1 whole number—1—has exactly 1 array.

These generalizations open the way to further work with prime and composite numbers. Leave room on the chart to record other arrays, perhaps for all sets up to and including 30 chips. Children can then list all the prime numbers between 1 and 31. By the time children complete this activity, they should be able to give a suitable definition of prime and composite numbers: "A prime number has only 2 different whole-number factors: 1 and the number itself." "A composite number has 3 or more different whole-number factors." The special characteristics of the number 1 should be stressed: it has only one factor—itself—and it is a factor of every other number. Make it clear that it is a special number that is neither prime nor composite. It is treated as a *unit*.

Work with prime and composite numbers extends understanding of factors, divisors, and multiples. Children have already encountered these terms in connection with their earlier study of multiplication and division. Now they should learn that the terms *factor* and *divisor* can be used interchangeably. When 2 whole numbers are multiplied, they yield a product; they can be called either factors or divisors of their product. (An exception to this statement is that 0 can be a factor but not a divisor.) The product of a pair of numbers can also be called a multiple of the 2 numbers. Thus 5 and 7 are factors or divisors of 35; 35 is a multiple of both 5 and 7.

The Sieve of Eratosthenes Another activity for intermediate grade children is an investigation of the sieve of Eratosthenes (see Activity 7-12). Eratosthenes was a Greek astronomer-geographer who lived in the third century B.C. He devised a scheme for separating any set of consecutive whole numbers larger than 1 into prime and composite numbers. The set of numbers most commonly used in classrooms is the set 1 through 100.

Boys and girls who are interested in further study of prime and composite numbers can complete these activities:

- Determine the prime numbers between 100 and 200, 100 and 300, or within any other limits, using the sieve to find the multiples of increasingly larger prime numbers—11, 13, 17, 19, 23, . . .
- Look for *twin primes*. Twin primes are two prime numbers that have exactly one composite number between them—3 and 5, 5 and 7, 11 and 13 are examples.

ACTIVITY 7-12 Sieve of Eratosthenes

Begin with a blank hundreds chart.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Put a ring around 1.

Two is the smallest prime number. Beginning with 4, mark out all of the multiples of 2 by putting a line through them.

The next prime number is 3. Ask "Which of the multiples of 3 have already been marked out?" (Those that are multiples of 2.) Mark out the remaining multiples of 3.

Ask, "Why are the 4 and its multiples already marked out?" (They are all multiples of 2.)

Ask, "Is 5 a prime number?" (Yes.) Mark out the multiples of 5 that remain.

Ask, "What is the smallest multiple of 7 that still remains?" (49.) Mark it out along with the remaining multiples of 7.

Eleven is the next prime number. Ask, "What multiples of 11 are smaller than 100? (22, 33, 44, 55, 66, 77, 88, 99.) "Are any of these left to be marked out?" (No.) "Why do you suppose no multiples of 11 are left?" (They are all multiples of smaller prime numbers.)

①	2	3	~~4~~	5	~~6~~	7	~~8~~	~~9~~	~~10~~
11	~~12~~	13	~~14~~	~~15~~	~~16~~	17	~~18~~	19	~~20~~
~~21~~	~~22~~	23	~~24~~	~~25~~	~~26~~	~~27~~	~~28~~	29	~~30~~
31	~~32~~	~~33~~	~~34~~	~~35~~	~~36~~	37	~~38~~	~~39~~	~~40~~
41	~~42~~	43	~~44~~	~~45~~	~~46~~	47	~~48~~	~~49~~	~~50~~
~~51~~	~~52~~	53	~~54~~	~~55~~	~~56~~	~~57~~	~~58~~	59	~~60~~
61	~~62~~	~~63~~	~~64~~	~~65~~	~~66~~	67	~~68~~	~~69~~	~~70~~
71	~~72~~	73	~~74~~	~~75~~	~~76~~	~~77~~	~~78~~	79	~~80~~
~~81~~	~~82~~	83	~~84~~	~~85~~	~~86~~	~~87~~	~~88~~	89	~~90~~
~~91~~	~~92~~	~~93~~	~~94~~	~~95~~	~~96~~	97	~~98~~	~~99~~	~~100~~

Point out to the children that the numbers that have not been marked out are prime numbers. Make a list of these numbers.

Many problem-solving opportunities arise during a study of number theory. Interested children in grades five and six can use calculators to simplify computation as they work with these problems:

PROBLEM 7-1

Prime and Composite Numbers

Put numbers such as these on three-by-five-inch cards:

421 221 161 391 131 247 539 457

Challenge children to determine which numbers are prime and which are composite, and stack them in 2 piles according to type.

PROBLEM 7-2

Testing Goldbach's Conjecture

Goldbach was a Russian mathematician who made a conjecture in 1742 that every even number greater than 2 can be written as the sum of a pair of prime numbers: $4 = 2 + 2$, $6 = 3 + 3$, $8 = 5 + 3$, $10 = 3 + 7$ or $5 + 5$. This conjecture has never been proven true or false for all even numbers. Challenge children to see if they can find an even number less than 500 that is not the sum of 2 prime numbers. (Children will need to determine prime numbers greater than 100 to do this problem.)

PROBLEM 7-3

Studying Prime and Composite Numbers

Have children finish a chart similar to the one at the top of the next page for the numbers 1–50:

Give them a sheet of paper that contains these questions:

1. Which numbers have 2 and only 2 factors? What kinds of numbers are these?
2. Which numbers have 3 and only 3 factors? What will be the next number that has exactly 3 factors?

Number	Factors	Sum of Factors
1	1	1
2	1, 2	3
3	1, 3	4
4	1, 2, 4	7
5	1, 5	6
6	1, 2, 3, 6	12
7	1, 7	8
8	1, 2, 4, 8	15
9	1, 3, 9	13
10	1, 2, 5, 10	18
11	1, 11	12
12	1, 2, 3, 4, 6, 12	28
13	1, 13	14
14	1, 2, 7, 14	24

3. Which numbers have 4 and only 4 factors? The numbers 8 and 27 have exactly 4 factors. How do they differ from the other numbers that have exactly 4 factors? Can you predict the next number of this kind? What number with exactly 4 factors follows 46?
4. Which numbers have 5 and only 5 factors? Can you describe these numbers? What is the next number that has exactly 5 factors?
5. Can you write a mathematical sentence that tells how to determine the sum of factors when a number is prime? Can you write a sentence for finding the sum when a number has exactly 3 factors?

Discuss with the children the results of their work.

INTEGERS

The numbers considered so far are used for counting discrete quantities or determining measures of magnitude such as length and volume. There are situations for which these numbers alone are not suitable. For example, measures such as temperature, time, distance, and altitude, and balances in money accounts require numbers that indicate both magnitude and direction. The numbers that do this are the real numbers. Integers are the part of the set of real numbers that are studied by elementary-school children.

Even though children have limited encounters with situations that require the use of integers, most have had some experience with them by the time they have completed the sixth grade. When primary-grade children learn to use thermometers, they will note that thermometers register temperatures below zero. Older children learn about altitudes above and below sea level, experience or read about gains and losses of yardage in football games, and experience gains and losses in money situations.

The study of integers in the elementary school is largely informal, with children having opportunities to use gamelike activities, stories, and number lines. Stories illustrated with simple drawings are an excellent way to introduce integers. The subject of these stories may be real or fanciful as long as the sequence of ideas can be readily understood. They should end in a way that leads to a discussion of conclusions that enhance children's understanding of these numbers. The following story, accompanied by simple pictures drawn on the chalkboard or projected with an overhead projector, fits all these requirements.

ACTIVITY 7-13 Positive and Negative Numbers

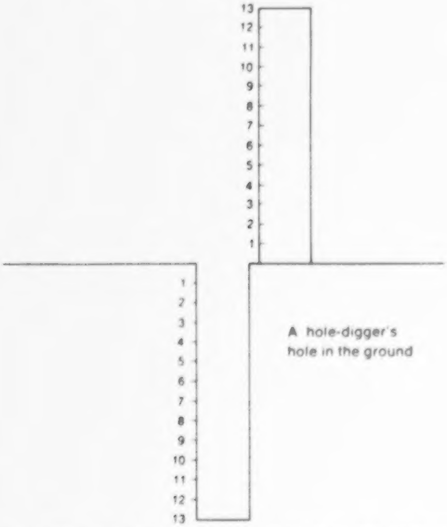
Read this story:

The Little Hole-Diggers

There was once a group of little hole-diggers. No one is sure how they looked, but some things are certain: the only time the hole-diggers got any joy out of life was when they were digging holes in the ground. They usually made very special holes that gave them very special pleasure. These special holes were straight down into the ground or straight into the side of a mountain, like this. (Display a drawing similar to that included here.)

Not only did the hole-diggers enjoy digging these holes, but they got extra special pleasure out of disposing of the dirt in a particular way. When they dug a hole straight down, the dirt from the first foot of the hole was placed right beside the hole in a pile 1 foot high. The dirt from the second foot of the hole was piled on top of the first foot of dirt, and so on. Also, they always marked the sides of their holes to show how deep they had been dug and marked the sides of their dirt piles to show how high they had been made. They dug holes into the sides of mountains in about the same way, except that they put the dirt on the ground in front of the hole, as you can see in the picture.

Ask questions: "Which numbers tell how deep the hole is?" "Which numbers tell how high the pile is?" "If we used the same numbers for both the hole and the pile, we might find ourselves getting mixed up as we talk about the hole and the pile of dirt. Can you think of how we could name each of these numbers so we won't get mixed up as we talk about them?" Children might use *above* and *below* for the vertical line


13
12
11
10
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8
7
6
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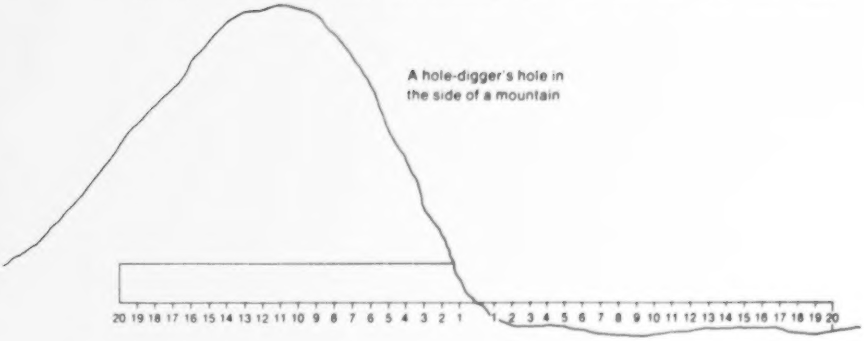
13
A hole-digger's hole in the ground


A hole-digger's hole in the side of a mountain
20 19 18 17 16 15 14 13 12 11 10 9 8 7 6 5 4 3 2 1
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

(hole) and *right* and *left* for the horizontal line (cave). Some may use *positive* and *negative*. Discuss each of the children's terms. If *positive* and *negative* are not given, you should introduce them, relating *positive* to *above* and *right*, and *negative* to *below* and *left*. Put negative signs in front of negative numbers and positive signs before the positive numbers.

Use the story situation and drawings to familiarize children with the uses of positive and negative numbers. Have a child locate negative 3 (-3) in the hole and then in the tunnel. Do this with both positive and negative numbers until children can readily locate any number.

Once children have learned the terminology and symbolism, they should practice using integers. David Page uses cricket jumps, and Louis

Cohen uses postman stories, as settings for introducing integers.[15] You can adapt their models to make games for practice. A sample of a game for each of these models follows:

GAME 7-5

Cricket Jumps

Materials: A number line on calculator tape extending from −12 to +12, construction paper crickets for markers, and a pair of dice, one marked −1, −2, −3, −4, −5, and −6; and the other marked +1, +2, +3, +4, +5, and +6.
Players: 2–4
Order of Play: Players roll the plus die, with highest number going first, next highest second, and so on.
Rules: The first player rolls both dice and reads the number on each one. From 0, the player moves a cricket in the positive direction by the number of jumps indicated by the plus die; and then in the negative direction by the number of jumps indicated by the minus die. Each player takes a turn in the same way. Play continues until a player is off the board at either end, or until a predetermined number of moves have been taken. The player farthest from 0 in either direction is the winner.

GAME 7-6

Checks and Bills

Materials: Play money in $1, $5, $10, and $20 denominations and a deck of cards with statements such as: "You receive a rebate of $2.00 from Wrangler jeans." "You get paid $5.00 for yard work." "You pay $5.00 for 2 hours of time at Waterworld." "You buy a baseball for $6.00."
Players: 2–4
Order of Play: Each player draws a card from the deck. Order is determined by value of the cards, highest to lowest.
Rules: Each player begins with $50.00—5 bills of $1.00, 1 of $5.00, 2 of $10.00, and 1 of $20.00. A bank containing $100.00 is set up. Shuffle the cards and put them face down. The first player draws the top card and tells whether it is a "positive" card or a "negative" card and how much will be gained or lost. For example, "I have a negative card that will cost me $5.00." Players receive money from the bank for positive cards and

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[15] See David A. Page, *Number Lines, Functions, and Fundamental Topics* (New York: Macmillan, 1964), Chapter 6, for a discussion of cricket jumps; and Louis S. Cohen, "A Rationale in Working with Signed Numbers," *Arithmetic Teacher*, XII, No. 7 (November 1965), pp. 563–567 for postman stories.

pay money to the bank for negative cards. Play rotates until one player is out of money or until each player has had a predetermined number of plays. The winner is the player with the most money.

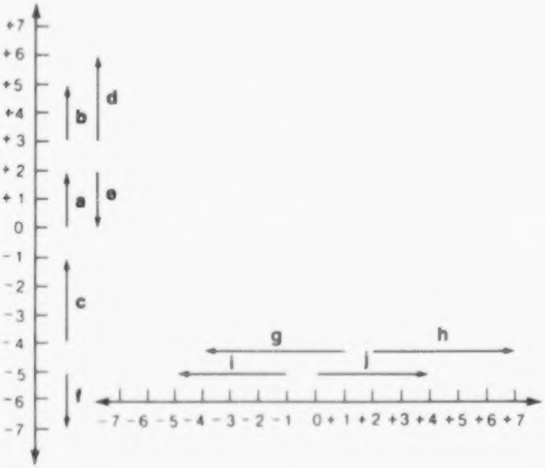
Vectors provide a good enrichment topic for interested children. Children should be shown how a vector, or directed segment, is a line segment with an arrow at one end to indicate direction on a number line. It represents an integer, or directed number.

ACTIVITY 7-14

Vectors

Show the vertical and horizontal number lines and point out the vectors on each one. Discuss the fact that each vector indicates both direction and magnitude. Relate the vectors to cricket jumps and money situations. "If a cricket jump is represented by vector **c**, where does the jump begin and end?" "If the vectors represent money situations, what does the vector **h** tell you?"

Be certain that the children become aware that a vector need not begin at 0.



Number lines can also be used to visualize the concepts of *equal to, greater than,* and *less than* with integers. On the whole-number line, the number to the right of another number is always the greater one; a number to the left of another number is always the smaller one. The same properties hold for numbers in the set of integers. When numbers are ordered on a number line, their comparative sizes can be determined by their locations. Zero and all positive numbers are greater than all negative numbers. To compare any two negative numbers, children should recognize that they are treated just like whole numbers. The number on the right is always the larger of the two numbers.

ENRICHMENT TOPICS FOR NUMERATION

Intermediate-grade children who demonstrate that they have a good knowledge of the Hindu–Arabic system can study early counting devices and historical and nondecimal numeration systems to extend their understanding of numeration. Examples of early counting devices are the quipu, or knots-on-a-cord system, used by the Inca Indians of Peru, the tally used in Europe until the early 1800s (Figure 7-14), abacuses such as the Chinese *suan-pan* (Figure 7-15), and the Japanese *saroban*.[16]

FIGURE 7-14

Simplified drawing of a tally stick showing a recorded debt of 234 pounds British money.

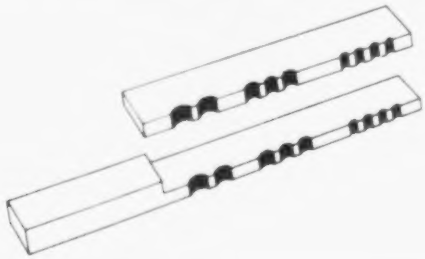
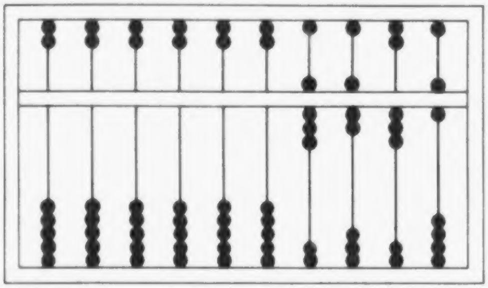


FIGURE 7-15

Chinese suan-pan *(abacus) representing the number 8736*



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[16] See Florian Cajori, *A History of Mathematical Notations, Vol. I* (LaSalle, Ill.: Open Court, 1928), pp. 30–40; and H. A. Freebury, *A History of Mathematics* (New York: Macmillan, 1961), p. 160 for information about these devices.

There are several historical numeration systems children can study. The Egyptian, Mayan, and Chinese offer interesting contrasts to our own. Any nondecimal system can be studied. Base-two is especially interesting to some children because of its use in computers. Children should have opportunities to share their findings with others through bulletin boards, articles in school newspapers, and demonstrations.

Summary

Children must have a well-developed understanding of numbers and the Hindu–Arabic numeration system if they are to be mathematically literate. Place-value devices, such as bundled tongue depressors, beansticks, Unifix cubes, base-ten blocks, and abacuses are useful for clarifying the meaning of the Hindu–Arabic system. Numerals can be written in both compact and expanded forms. Expanded forms are useful for helping children clarify and understand place value and algorithms for such operations as addition and subtraction. Regrouping is a process in which exchanges are made between adjacent place-value positions in a number during computation with some algorithms. Children need to understand regrouping thoroughly before they deal with algorithms. Beansticks and other place-value devices can be used to teach the process.

Children must understand one-to-many, many-to-one, and many-to-many correspondences, for which numerous applications are found in mathematics. Skip-counting is a useful process for counting money, vote tallies, and grouped items. Number theory topics contribute to children's understanding of numbers. Primary-grade children learn about odd and even numbers, while intermediate-grade students learn about prime and composite numbers. Children who are introduced to integers should learn about them through stories and activities that are meaningful to them. A study of early counting devices, historical numeration systems, and nondecimal systems can be used as a means of extending the knowledge of counting and numeration for children in the intermediate grades.

Study Questions and Activities

1. You should be familiar with a variety of materials for teaching the meaning of place value: beansticks, tongue depressors, place-value pocket charts, Unifix cubes, base-ten blocks, Cuisenaire rods, place-value chips, and the abacus. Show the meaning of the following numbers with each of the materials you have.

   (a) 136 (b) 95 (c) 308 (d) 4082

Make drawings to represent each number in 3 ways. Identify each of the materials and drawings as proportional or nonproportional.

2. Different forms of expanded notation are described in this chapter. Some forms are suitable for introducing place value to first graders: "1 ten and 6 ones." A form using exponents is introduced to intermediate students. Express each of the following numbers in 4 different expanded forms.

   (a) 654 (b) 47,825 (c) 200,409

   Begin with the simple form, then use 2 intermediate forms, and end with the exponential notation.

3. Create a set of skip-counting materials on index cards. For twos, you might cut out pairs of eyes or bicycles. For fours, use pictures of automobiles showing 4 tires. Hands are good for showing fives. Add these cards to your teaching file.

4. Daniel Hardin's article (see For Further Reading section) describes how notches in pig's ears can be used to acquaint children with the base-3 system of numeration. Read the article and explain how the ear-notching system is used. Draw pictures to show notches for the fifth pig in a year's third litter; the twelfth pig in a year's eighth litter.

5. Read "The Plight and Might of Number Seven," "Don't Be Blue, Number Two," or "Number Three Comes to See" (see For Further Reading section), and list some student activities suggested by Shmuel Avital. How might you use these activities in your class for extension and enrichment?

6. Prime number can be approached in several ways. Burton and Knifong (see For Further Reading section) discuss 6 methods. List the approaches and discuss their strengths and weaknesses. Which method appeals most to you? Why? Show that you understand the file card and toothpick activity discussed by the authors by picturing cards for these numbers: 2, 3, 4, 6, 7, 9.

7. Goldbach's conjecture holds that every even number can be expressed as the sum of 2 prime numbers. Hamann's conjecture suggests that every even number from 2 to 250 can be expressed as the *difference* of 2 prime numbers. Read more about Hamann's conjecture in Maxine Frame's article (see For Further Reading section). Test Hamann's conjecture with at least 20 numbers between 2 and 250 to demonstrate that it is reasonable.

8. Check several back issues of the *Arithmetic Teacher* for activities. Sections entitled "IDEAS" and "From the File" are excellent sources of ideas for many topics. The first section contains suggestions for lessons

on 1 topic for several grades, and the second section contains simulated file cards, each presenting a problem, puzzle, teaching tip, or other information. Add copies of useful ideas to your teaching file.

For Further Reading

Anderson, Alfred L. "Why the Continuing Resistance to the Use of Counting Sticks?" *Arithmetic Teacher*, XXV, No. 6 (March 1978), p. 18. Children who use counting sticks (Popsicle sticks) in conjunction with abstract numbers and pictorial representations in textbooks and work sheets performed with more understanding than those restricted to workbooks, work sheets, finger counting, and other methods of tallying.

Avital, Shmuel. "Don't Be Blue, Number Two," *Arithmetic Teacher*, XXXIV, No. 1 (September 1986), pp. 42–45. Activities that illustrate interesting characteristics of the number 2.

———. "The Plight and Might of Number Seven," *Arithmetic Teacher*, XXV, No. 5 (February 1978), pp. 22–24. The number 7 comes alive to visit the author and tells all about its special characteristics. Many interesting possibilities for student investigations of the number 7 arise from this article.

Avital, Shmuel, and Uri Grinblat. "Number Three Comes to See," *Arithmetic Teacher*, XXX, No. 7 (March 1983), pp. 46–49. Activities that focus on the number 3.

Brennan, Alison D. House. "Mr. Ten, You're Too Tall," *Arithmetic Teacher*, XXV, No. 4 (January 1978), pp. 20–22. A set of materials to help children make the transition from the conceptual stage to the computational stage of thought when performing operations with base-ten numbers is described. The materials are both proportional and nonproportional and can be used to introduce place-value concepts and then to teach computational processes.

Brumbaugh, Frederick L. "Big Numbers in a Classroom Model," *Arithmetic Teacher*, XXIX, No. 3 (November 1981), pp. 18–19. A model of the solar system, scaled to metric units and small enough to fit in a classroom, gives children practice in reading large numbers as they deal with distances and diameters.

Burton, Grace M. "Teaching the Most 'Basic' Basic," *Arithmetic Teacher*, XXXII, No. 1 (September 1984), pp. 20–25. Suggests activities for understanding the decimal number system—the most 'basic' basic.

Burton, Grace M., and J. Dan Knifong. "Definitions for Prime Numbers," *Arithmetic Teacher*, XXVII, No. 6 (February 1980), pp. 44–47. Materials and procedures for teaching children about prime numbers are discussed. Other activities dealing with prime numbers are suggested.

Cajori, Florian. *A History of Mathematical Notations, Vol. I* (LaSalle, Ill.: Open Court, 1928). The history of symbols and notation for many ancient systems is presented in Chapter 2. Of special interest is a section on fanciful hypotheses about the origins of the symbols for the Hindu–Arabic system.

———. *A History of Mathematics* (New York: Macmillan, 1919). In the early chapters, the history of many numeration systems and mathematics in the countries where the systems had their origin is presented; included are Babylonian, Egyptian, Greek, Roman, Mayan, and Hindu–Arabic numeration systems.

Frame, Maxine R. "Hamann's Conjecture," *Arithmetic Teacher*, XXIII, No. 1 (January 1976), pp. 34–35. "Hamann's conjecture" was made by a seventh-grade student when he discovered that every even number from 2 through 250 can be expressed as the difference between two prime numbers. (Later, when he did research about it, Hamann discovered that his conjecture was not the first expression of this idea.)

Freebury, H. A. *A History of Mathematics* (New York: Macmillan, 1961). In a rather brief but very interestingly written book, Freebury describes the origins of the Hindu–Arabic and other numeration systems. This book is especially recommended for the elementary-school teachers' library.

Hardin, Daniel D. "Teaching Base Three? In a Pig's Ear," *Arithmetic Teacher*, XXVII, No. 1 (September 1979), pp. 48–49. Hog breeders use a uniform system of notching pigs' ears to keep records of each pig's time of birth. The system employs a base of 3; it can be used to introduce the system to children.

Lappan, Glenda, Elizabeth Phillips, and M. J. Winter. "Powers and Patterns: Problem Solving with Calculators," *Arithmetic Teacher*, XXX, No. 2 (October 1982), pp. 42–44. Children use calculators to work with patterns that result from use of powers of numbers and patterns that are generated. Older children get much practice using exponents and reading large numbers.

Makurat, Phillip A. "A Look at a Million," *Arithmetic Teacher*, XXV, No. 3 (December 1977), p. 23. A computer printout of a million "$" signs is relatively easy to acquire—much easier than a million bottle caps. With such a printout, children can determine how long it would take each one to print a million dots. Other possible activities are suggested.

March, Rosemary. "Georges Cuisenaire and His Rainbow Rods," *Learning*, VI, No. 3 (November 1977), pp. 81–82, 84, 86. Gives a brief account of the man and how he became inspired to develop the Cuisenaire rods. Describes the struggle to gain acceptance of the rods and includes testimonials to their value.

Ronshausen, Nina L. "Introducing Place Value," *Arithmetic Teacher*, XXV, No. 4 (January 1978), pp. 38–40. Explains carefully sequenced steps for teaching place value. Although they were developed in one-to-one tutorial situations, the author claims the activities are suitable for small group work too.

Shokoohi, Gholam-Hossein. "Manipulative Devices for Teaching Place Value," *Arithmetic Teacher*, XXV, No. 6 (March 1978), pp. 48–51. Acorns and glasses provide the means for developing the place-value concept as children record the count of passing cars by one-to-one-correspondence. What to do when the first glass is full becomes the basis for determining that a second glass is needed, and the place-value concept is introduced.

Smith, David Eugene, *History of Mathematics, Vol. II* (Boston: Ginn and Company, 1953). The second chapter presents excellent material about the history of numeration systems.

———, and Jekuthial Ginsburg. *Numbers and Numerals* (Washington, D.C.: National Council of Teachers of Mathematics, 1937). This is one of the early

publications of the National Council; it is a highly readable account of numeration, including its history, suitable for older children as well as teachers.

West, Mike, and Ken Haar. "It's Neat Being Surrounded by Peanuts," *Arithmetic Teacher*, XXVI, No. 1 (September 1978), p. 22. Mike, a ten-year-old student, bought a 100-pound bag of peanuts and held a contest; fellow students guessed the number of peanuts. To determine the number, Mike and a few friends counted the peanuts, grouping by tens and powers of tens—there were 16,870. Guesses ranged from as low as 800 to well over a billion, prompting Ken, the teacher, to observe that even some high-school students who made estimates lack good number sense.

Chapter

8

Informal Geometry

In Chapter 8, you will read about

1. the importance of geometry in elementary mathematics
2. the three stages of geometric understanding outlined by van Heile that describe elementary-school children
3. topological geometric concepts and activities that provide a foundation for later geometry
4. concrete materials and activities to introduce plane and solid figures and their characteristics
5. Logo as an approach to exploring geometry
6. developing concepts of congruence, similarity, and symmetry with concrete materials
7. introducing coordinate geometry
8. adapting geometry instruction to the needs of special learners

Geometry has been part of the elementary-school curriculum for many years, and children have learned to recognize and name common plane and space figures. Today geometry enjoys a larger share of the time devoted to mathematics because educators recognize that a more comprehensive study of the subject is beneficial to children.

Knowledge of geometry enhances children's understanding of their world. Kindergarten and primary-grade children who develop good spatial abilities through activities dealing with proximity, separation, order, and enclosure can comprehend their visual surroundings better than children whose spatial abilities are less developed. Many misunderstandings of pictures and diagrams used to develop important mathematical concepts are the result of young children's poor spatial abilities. Intermediate-grade children should learn and appreciate both practical and aesthetic reasons for geometric forms. Geometry activities contribute to their development of knowledge and appreciation of their environment.

Geometry contributes to children's problem-solving skills. There are many possibilities at each grade level for engaging children in problem-solving activities. A number of activities that follow require that children investigate geometric forms, search for patterns, prepare and interpret tables, and make generalizations.

Activities in geometry provide a change of pace for children. They allow children to get away from numbers and operations from time to time. Such breaks are psychologically good for children because they give a mathematics program variety and help prevent development of negative attitudes. Many children regard geometry as a "fun" topic and eagerly look forward to activities dealing with the subject.

The mental development of students with respect to geometry has been studied by P. van Heile, a Dutch mathematician. He has concluded that individuals pass through 5 stages of development as they learn the subject. The first 3 stages will concern you as a teacher of elementary-school children. The first stage occurs before children's entry into school and during the kindergarten and early primary grades. During these years, children learn to recognize and name common plane figures, such as squares, circles and rectangles, and simple solids, such as cubes and spheres (balls). In the next few years, children pass through the second stage as they learn the general characteristics of figures and are able to specify the number of sides and angles in a square or triangle and the number of faces in a cube. By the end of grade six, children are in stage three, where they are able to classify shapes according to their characteristics. Now they recognize that all four-sided figures are quadrilaterals and that a solid figure formed by congruent faces is a Platonic solid. Your expectations for children as they study geometry and the activities you provide in your classroom should be developed with these stages in mind. Your program must be informal so that children can move through the 3 levels with an understanding of the geometry they learn. A formal, deductive study of geometry, which is van Heile's fourth stage of development, must be delayed until students are mature enough to cope with it.[1]

A comprehensive discussion of materials and procedures for teaching geometry is beyond the scope of this book. The topics presented in this chapter are common to most elementary-school programs. The materials and procedures are typical of those in schools where an informal, concrete-manipulative approach to teaching the subject is a part of the mathematics program. The activities advocated here will supplant those in primary-grade workbooks and intermediate-grade textbooks. If books are used, it is for the practice and reinforcement materials they contain rather than as the basic source of lesson content.

Before presenting information about teaching geometry, it will be useful to review Piaget's stages of mental development and the implications of his research on the teaching of geometry.

TOPOLOGICAL VERSUS EUCLIDEAN CONCEPTS OF SPACE

You will recall that Piaget believed that each individual progresses through 4 stages of mental development during maturation:

1. The sensorimotor stage (0 to 2 years).
2. The preoperational stage (2 to 7 years).
3. The concrete operations stage (7 to 11 years).
4. The formal operations stage (11 years and older).

Piaget said that children's first concepts of the world around them (space) are topological rather than Euclidean. A topological view does not require that figures maintain a rigid or fixed shape, which is the Euclidean view. Rather, a shape may be altered so that it assumes a new configuration, much as it would if drawn on a sheet of rubber where it can be pulled in different directions. For example, a square may assume the shape of a rectangle or an oval or some other simple closed curve. An open figure, such as the letter *C*, can be reshaped to look like the letter *S*. As the configurations change, there are certain characteristics that remain unchanged. The square remains a simple closed figure, and the open figure remains open with two end points.

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[1] For a discussion of van Heile's 5 levels, see Izaak Wirzup, "Breakthroughs in the Psychology of Learning and Teaching Geometry," in *Space and Geometry*, J. Larry Martin, ed. (Columbus, Ohio: ERIC Information Analysis Center for Science, Mathematics, and Environmental Education, 1976), pp. 76–79.

According to Piaget, very young children do not view people and objects as being stable and unchanging but view them in the topological sense. To children in the sensorimotor stage, the appearances of people and objects change as the positions from which they are viewed change. This view of the world continues into the preoperational stage of development. Teachers who are aware of how young children view the world do not expect that their students will be exact in the manner in which they reproduce figures during activities with plane and space figures. Neither will they deal with topics that are beyond the level of children's maturity. As children move from the preoperational to the concrete operational stage, their study of geometry can change from topological to Euclidean.

There are four topological (spatial) relations—proximity, separation, order, and enclosure—that are suitable for kindergarten and primary-grade children. These and other early geometry activities should be introduced at the same time as and be presented simultaneously with activities designed to develop children's understanding of numbers and numeration.

Proximity

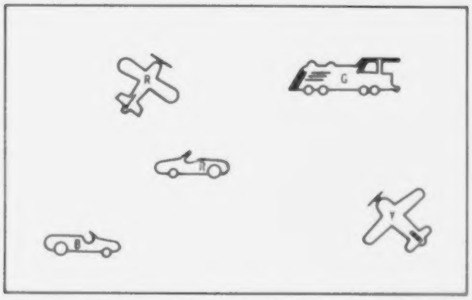
Proximity refers to the nearness of one object to another. Naturally, very young children are interested in things near to them, because they can touch and manipulate them. Things that are out of their reach are usually of little interest, unless the child sees something that is eye-catching such as a shiny part of a swinging mobile. Objects that are out of sight do not exist in the mind of a child in the early sensorimotor stage.

Gradually children engage in activities that help them recognize that out-of-sight objects exist and clarify the distinction between near and far and the relationships of objects in terms of their nearness to each other. The prenumber activities discussed in Chapter 6, particularly the ones dealing with classification and spatial relationships, provide experiences that help develop children's understanding of proximity. Once children have classified a set of materials or arranged a set of beads to form a pattern on a string, you can ask such questions as: "Which red car is nearest to the green car?" "Which black bead is farther from the blue bead?"

ACTIVITY 8-1

Near and Far

Set up a flannel board that displays colored felt cars, airplanes, and locomotives.



- Ask questions like these:
  - ☐ "Which is nearer the red airplane, the red car or the green locomotive?"
  - ☐ "Which object is farthest from the green locomotive?"
  - ☐ "Sally, will you move the blue car so that it is nearer to the locomotive than it is to the yellow airplane?"
- Variations: Use magnetized objects on a magnetic board. Use objects such as apples, oranges, and bananas placed on a tabletop. Have children go to places in the room that are near or far from tables, cabinets, doors, and so on.

A playground offers many opportunities for dealing with proximity on a larger scale than with a flannel board in the classroom. Take a class to the school yard and give instructions like these:

- Have an individual or a small group of children stand near a set of monkey bars.
- Arrange a group of children in a line between a basketball hoop and a tetherball pole. Ask questions: "Who is standing nearest to the basketball hoop?" "Who is farthest from the hoop?" "Is Sally nearer the hoop or the pole?" "Who is about the same distance from the pole as from the hoop?"
- Point to 2 objects, such as a red rosebush and an elm tree. "Which is farther away (or nearer), the rosebush or the tree?" "Is it farther to the fence over there, or to the parking lot over there?"
- Ask questions that cause the children to think about reasons for the location of certain areas of the school ground: "Why is the parking lot near the school buildings and not at the other side of the playground?" "Why is the intermediate-grade playground over there, while your playground is up here?"

Separation

Until children achieve an understanding of separation, they cannot clearly visualize an object as having separate parts or a collection as being made up of separate objects. All of the parts run together in the child's mind. Children's drawings provide a good indication of their understanding of separation. Drawings of human figures are particularly enlightening. A child lacking the concept of separation is likely to draw a figure on which the arms and legs are attached directly to a bodiless head or are otherwise distorted. A child who understands separation will put the arms and legs in place with the body properly separating them, and the eyes will be in their proper position on the face with the nose separating them.

ACTIVITY 8-2

Separation

- Arrange 2 sets of felt shapes about 6 inches apart on a flannel board with a piece of yarn stretched from top to bottom separating them.
- Have the children describe what they see. During the discussion, say that the yarn separates 1 set from the other.
- Move the sets about, putting them closer together or farther apart, with the yarn still between them. After each move, ask if the yarn still separates the sets.
- Move both sets to the same side of the yarn. "Does the yarn still separate the 2 sets?"

Again, a trip to the school grounds will give opportunities to discuss separation. Here are typical situations you can use:

- A fence separates a playground from a street adjacent to it.
- A red line separates an intermediate-grade playground from a primary-grade playground.
- The office complex separates the kindergarten–primary wing from the intermediate-grade wing (This gives you an opportunity to point out that a line is not the only thing that can separate 2 things.)
- Children can also discuss the fact that rivers, freeways, railroad tracks, and similar natural or human-made objects can separate one part of their community from another.

3 Order

Activities dealing with continuous and discrete objects, classification, and patterns, such as the ones described in Chapter 6, contribute to children's understanding of order. You might want to review activities 6-8, 6-12, 6-13, and 6-14 to refresh your memory about suitable free-play and teacher-directed activities dealing with order. During early work in this area, children who can copy a pattern directly from a model may be unable to reproduce the pattern in reverse order. They will be able to do this if they are given enough unhurried teacher-directed and problem-card activities.

ACTIVITY 8-3

Balls in a Tube

- Use rubber or plastic balls of 3 different colors and a tube through which the balls will freely roll (a paper-towel roll will do). Color the tube red at one end and green at the other end.
- As the children watch, put a red ball into the tube at the red end. Next put in a blue ball, followed by a yellow ball. Ask: "Which ball will come out of the green end first?" "What will be the color of the ball that comes out after the red ball?" "What color will the last ball be?" Repeat with different sequences of colors.
- Repeat the red-blue-yellow order. This time, tell the children, "I am going to turn the tube around so that the green end is where the red end is and the red end is where the green end is." Do that. "Now what will be the color of the ball that comes out of the green end first?" "What will be the color of the ball that follows the first one out?" "What color will the last ball be?" Repeat this with different orders of colors. Also, after putting balls into the tube through the red end and reversing it, ask the children to name the order of colors before the balls are taken out at the red end.

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4 Enclosure

Enclosure includes the positioning of 1 point between 2 others on a line, a point within a closed curve on a plane, and a point within a closed space figure. Enclosure on a line is important because it is a relation children encounter frequently in mathematics. For example, they will talk about a number being between 2 other numbers. (A number that is between 2 numbers can be said to be enclosed by the numbers.) Children's first work with enclosure should be at the concrete-operations level. Activity 8-4 is for developing children's understanding of enclosure on a line.

ACTIVITY 8-4

Enclosure on a Line

Put 3 different beads on a string.

Have children describe what they see. If they have already considered separation, they can mention that the center bead separates the other 2.

Point out that the 2 outside beads enclose the bead in the middle. Help them see that the middle bead cannot be taken off without removing 1 of the 2 outside beads.

Variations: Have children stand so that one is enclosed by others on either side. Use a play clothesline and discuss items of clothing that are enclosed by other clothing on the line. Discuss how some chairs are enclosed by others in a row of chairs.

Toys and pictures can be used to develop children's understanding of enclosure on a plane. The rural setting in the following activity is one way to use toys.

ACTIVITY 8-5

Enclosed by a Fence

Set up a scene with plastic fences, horses, cows, barns, and so on. Place some animals inside a closed fence and some outside of it.

Ask questions similar to these:

- ☐ "Who can show me some animals that are inside the fence?"
- ☐ "Is the brown horse inside or outside of the fence?"
- ☐ "Is the barn inside or outside of the fence?"

Point out that the things inside the fence are enclosed by the fence. As long as the gate is closed, they cannot get out, and the animals that are outside cannot get in.

Variations: (1) Felt animals and a yarn loop can be used on a flannel board. (2) Playgrounds for kindergarten children are frequently enclosed by a fence or a fence and building wall. Discuss the fact that some equipment, such as a set of climbing bars, are enclosed by the fence. Other equipment, such as wagons, and the children themselves can be moved into or outside of the enclosure. Discuss other pieces of equipment in terms of whether or not they are permanently enclosed by the fence.

Have a wide assortment of boxes and bottles that are opaque and clear to build children's understanding of enclosure in space. An activity like this can be used:

ACTIVITY 8-6

Enclosed in a Jar

Put a colored marble or small rubber ball in a clear jar with a lid and another one in an open jar.

Have the children explain the differences between the 2 bottles.

Ask: "Can I take the marble out of the jar with the lid without removing the lid?" "Can I take the marble out of the other jar without doing anything to the jar?"

Discuss the fact that the closed jar encloses the marble so that it cannot be taken out and other marbles cannot be put in without removing the lid.

Put a cotton ball in a lidded opaque box.

Shake the box; then ask, "Do you think the cotton ball is still in the box?" "Does the box enclose the cotton ball?"

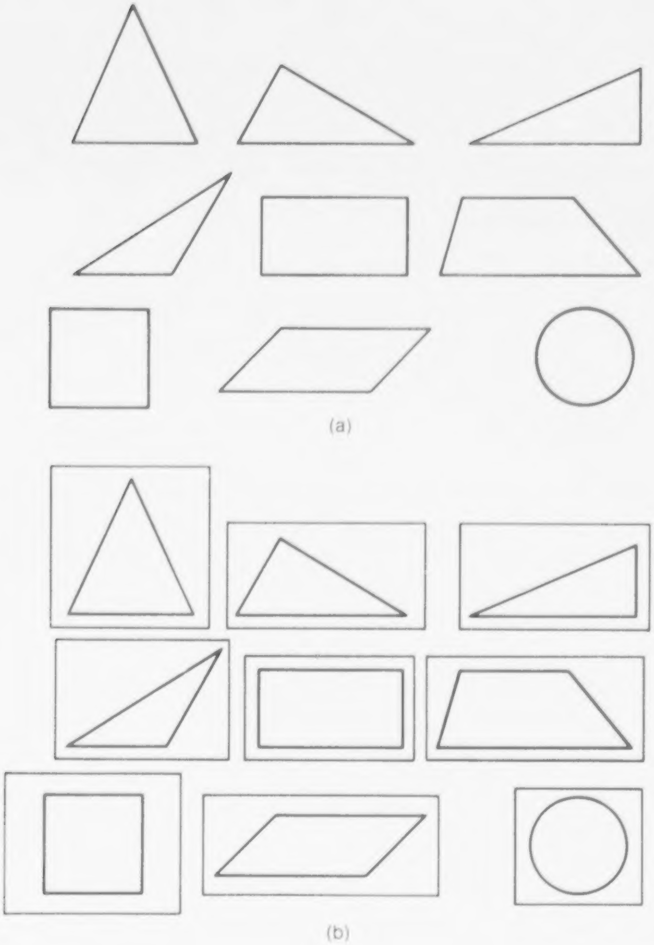
Repeat with other objects in both open and closed boxes and bottles. Be sure children understand that when an object is enclosed in a box or bottle, it cannot be taken out without opening the container.

ACTIVITIES WITH PLANE FIGURES

Plane figures, including circles and closed figures composed of line segments—triangles, squares, and rectangles—are commonly introduced in the first grade. In each succeeding grade, children study these and other figures more thoroughly until by the time they leave the sixth grade they can classify figures by their common characteristics, or attributes. The activities that follow are based on the belief that children's experiences should be at the concrete-manipulative mode of learning. The activities for each material are not exhaustive but are representative of things children can do with the material.

FIGURE 8-1

(a) Common geometric shapes. (b) Posting holes for the shapes.



Shapes and Posting Holes

Shapes cut from plywood, masonite, or heavy art paper such as colored railroad board are easily made. If they are cut carefully, you will have not only the shapes, as shown in Figure 8-1(a), but also their outlines in the piece of material from which they were cut (b). These holes are called posting holes. Children use the models to get the "feel" of each shape by running their fingers around the outside of the shapes and the inside of their posting holes. Have children count the sides (edges) and corners of each shape and talk about the way corners feel: they will feel "sort of flat," "sharp," or "square." Children can also talk about the differences in the feel of the edges of polygons compared with a circle.

Once they are familiar with some shapes, primary-grade children should search their environment for examples of the ones they know. Activities such as the following are interesting to most children:

> Children can look around the classroom to find things that have shapes similar to the geometric figures they have been studying. Many of the figures will be represented by objects already in the room. You should prearrange the room to contain representative objects of certain shapes not usually present, particularly spheres and circles.
>
> During their study of safety, children can be guided to recognize the different shapes of signs used along streets and highways (Figure 8-2). Replicas or pictures of them should be available.
>
> Children can look in magazines for pictures of objects of different geometric shapes. The pictures can be displayed on bulletin boards.

Later, fifth and sixth graders can use the cut-out models and posting holes to learn more advanced geometric ideas. An investigation that uses these materials to learn about symmetry is discussed later.

Geoboards

A geoboard is a square piece of wood or plastic in which pegs are arranged in some orderly fashion. Arrangements in common use have 9, 16, 25, or 36

FIGURE 8-2

Common highway signs that children should become acquainted with by shapes and colors

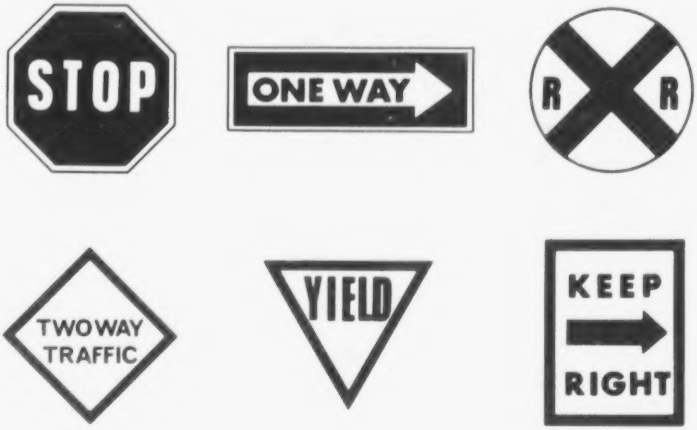
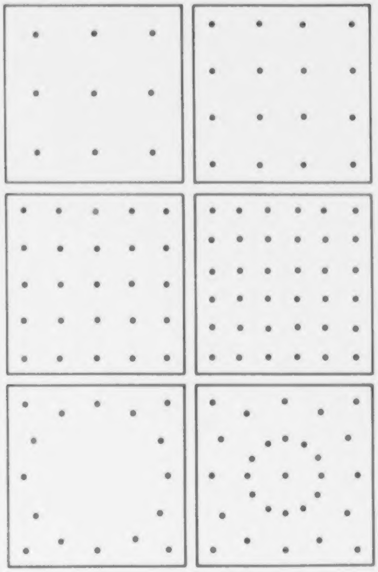


FIGURE 8-3

Examples of commonly used geoboards



pegs ordered in 3-by-3, 4-by-4, 5-by-5, and 6-by-6 patterns, and some have circular patterns (Figure 8-3). Boards with 100 pegs arranged in a 10-by-10 pattern are also useful because they have room for the outlines of several plane figures at one time. Rubber bands are used to make figures on the boards. Paper covered with rows and columns of evenly spaced dots should be available so children can copy figures from their geoboards. Geoboards and dot paper are produced commercially by a number of companies. Many teachers make geoboards with plywood and small finishing nails, and dot paper with the copy machine.

Geoboards make possible a wide range of investigations. In the primary grades, children can use them to investigate and gain an intuitive understanding of plane figures, both open and closed, and of points and line segments. Older children can use them to study different types of triangles, quadrilaterals, and other figures and to learn about regular and irregular figures, concave and convex figures, and other more mature concepts.

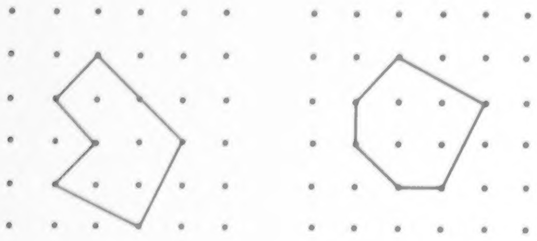
ACTIVITY 8-7 Three-Sided Figures

- Have children work singly or in pairs.
- Show them how to make a 3-sided figure with one rubber band.
- Have the children compare their triangles.

- Ask: "Are all of the figures alike?"
- Discuss the ways any of the figures are different.
- If the term *triangle* is unknown, introduce it now. Tell the children that each of the figures they made is a triangle.
- Repeat with 4-sided figures made with 1 rubber band.

ACTIVITY 8-8 Polygons

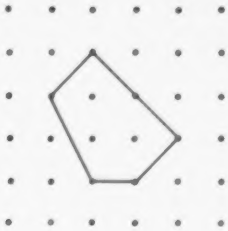
- Have children make a series of figures, each one having 1 more side than its predecessor. (Ten-by-ten boards are good for this activity. If smaller ones are used, children will probably need to copy each figure on dot paper before a new one is made. Do not do this activity unless your children are mature enough to make accurate copies.)
- Have the children compare their figures with those of their neighbor to see the differences, if any, in each kind.
- Talk about changes in the figure's shapes as more sides are used.
- Some children will probably make concave as well as convex figures. The figure on the left in the illustration is concave, and the one on the right is convex. Accept these as long as children recognize that they are closed figures and have the number of sides claimed by their makers.



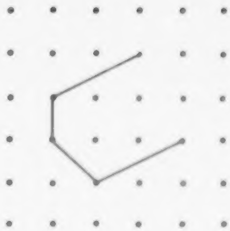
(a) (b)

ACTIVITY 8-9 Open and Closed Curves

- Have children make open and closed curves. Open curves can be made by stretching a rubber band between 2 or more pegs, either in a straight line or going around a corner.



(a)



(b)

Discuss differences in the children's figures.

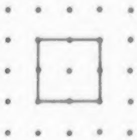
Extend children's understanding of enclosure by considering pegs located inside both closed and open figures. Discuss the fact that another rubber band cannot be stretched between a peg inside the closed figure and 1 outside the figure without crossing the figure itself.

There are geoboard activities to help older children extend their knowledge of plane figures to van Heile's third level of knowledge. The ones that follow are examples of some you can give orally or present with problem cards.

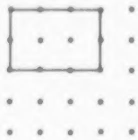
ACTIVITY 8-10

Four-Sided Figures

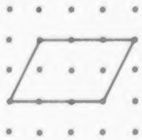
Have children make these 4-sided figures on their geoboards:



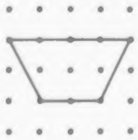
Square



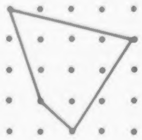
Rectangle



Parallelogram



Trapezoid

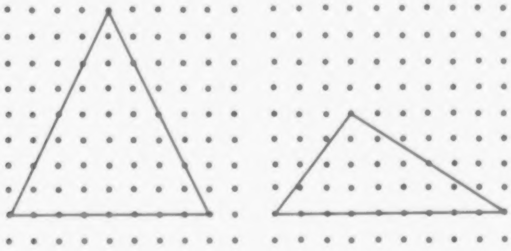


Trapezium

Discuss each figure to point out the similarities and differences.

Organize information about each figure in a table.

Figure	Angles	Sides
Square	four right (90°)	Congruent, parallel
Rectangle	four right (90°)	Opposite sides congruent and parallel
Parallelogram	Opposite angles congruent	Opposite sides congruent and parallel
Trapezoid	May have adjacent angles congruent	One pair of parallel sides, nonparallel sides may be congruent
Trapezium	May have two adjacent angles congruent	No parallel sides, may have two adjacent sides congruent



Isosceles Scalene

FIGURE 8-4

Two triangles classified by the characteristics of their sides shown on a 10-by-10 geoboard

Models of some triangles classified by their sides can be made on larger geoboards, such as 10-by-10 boards (Figure 8-4). During a study of these models, children will learn to classify triangles as isosceles, which have 2 congruent sides, and scalene, which have no congruent sides. Children can also learn that triangles are classified according to the types of angles they have, as shown in Figure 8-5. In (a) the 3 types of triangles classified by angles are illustrated. A right triangle has one right angle; an acute triangle has 3 acute angles; an obtuse triangle has 1 obtuse angle, which is an angle that is greater than 90 degrees. These models can also be used to illustrate the meaning of a triangle's altitude (b). For the right and acute triangles, a rubber band (shown by the broken line and representing the altitude) is inside of the triangle; it is outside of the figure when the triangle is obtuse. (If the obtuse triangle is placed with the longest side at the bottom, a different altitude is shown by a broken line in its interior.)

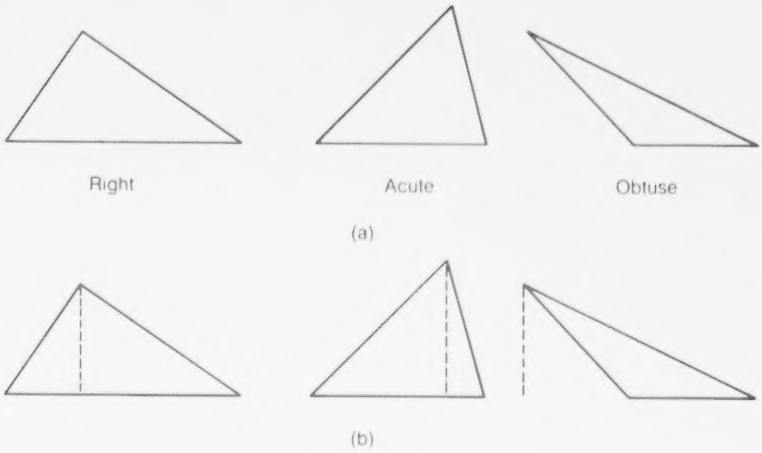


FIGURE 8-5

Three triangles classified by characteristics of their angles (a) and their altitudes (b).

Other Activities

By the end of sixth grade, children should be familiar with the common figures shown in Figure 8-6. They should also be able to explain how each closed figure separates a plane into 3 distinct sets of points—the set that is the polygon itself, the set within the polygon (enclosed by the polygon), and the set outside the polygon. When children find the area of a region, they actually determine the measure of that part of the plane enclosed by the simple closed curve. When they find the perimeter, they determine the measure of the line segments (or the curved line) that comprise a simple closed curve.

The final activity is one for which children must use several problem-solving strategies. They can make shapes of common plane figures and show the number of diagonals for each on geoboards. A diagonal is a segment that connects 2 nonadjacent corners. Then they should organize their data in a table.

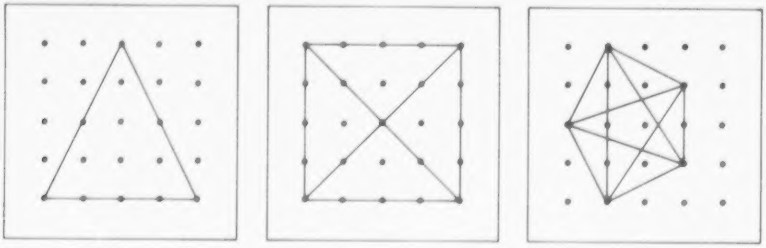
FIGURE 8-6

Chart to classify polygons

Polygon	Name of Polygon	Number of sides	Name of Region
	Triangle	3	Triangular
	Quadrilateral	4	Quadrangular
	Pentagon	5	Pentagonal
	Hexagon	6	Hexagonal
	Heptagon	7	Heptagonal
	Octagon	8	Octagonal

PROBLEM 8-1 Diagonals

Have the children make each of these figures and show its diagonals on a geoboard.



State the name for each figure and count its diagonals. Put the data in a table.

Figure	Angles	Diagonals
Triangle	3	0
Quadrilateral	4	2
Pentagon	5	5
Hexagon	6	9
Heptagon	7	14

Examine the table to determine how the number of diagonals changes as the diagonals for each succeeding figure are made. Write the pattern on the chalkboard: 2, 3, 4, 5, ...

Have the children predict the number of diagonals in an 8-sided and a 9-sided figure. Check their predictions by making 8- and 9-sided figures and their diagonals.

Challenge children to determine a formula for finding the number of diagonals in an *n*-sided figure.

Trap-a-zoid[2] is a geometry game on the computer that goes beyond memorization of shape names. Children capture "Zoids" by drawing geometric shapes around them. Geometric properties are learned by plotting lines and points on the screen.

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[2] Collamore Educational Publishing, 125 Spring St., Lexington, MA 02173.

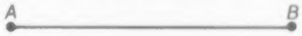
ACTIVITIES WITH POINTS AND LINES

The very nature of geometric points and lines makes them difficult for children to conceptualize. Piaget recommended that study of these figures be postponed until children are eight or nine years old, which means at least the third grade of elementary school. By this time most children will have a background of earlier experiences upon which to build their understanding. Once points and lines are introduced, children will study them and plane and solid figures in such a way that work with 1 type of figure will extend and reinforce their understanding of the others.

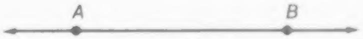
ACTIVITY 8-11 Segments and Rays

Hold a piece of string with 1 end in each hand; draw the string taut against the chalkboard. Tell the children that the string represents a part of a line that could go clear across the chalkboard. Discuss the fact that the line could extend forever in both directions from the board. Tell them that a part of a line is called a segment.

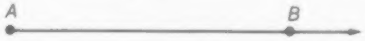
Draw a picture of a segment on the board that is as long as the taut string. (Carpenters use a string called a chalk line to mark straight lines on flat surfaces. A chalk line would be useful here because once the line has been held taut against a chalkboard, it can be snapped to mark a line.) Label it to show how line segments are named.



Draw a picture of a line on the board and show how it is labeled. Explain that the arrowheads at each end are used to show that the line goes on without end.



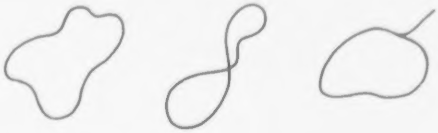
Draw a picture of a ray on the board. Explain that a ray includes an end point and all points extending in 1 direction from that point. Ask: "Why is there only 1 arrowhead on a ray?"



ACTIVITY 8-12 Closed Curves

Give each a child a 50-centimeter length of yarn.

Show a picture of each of these shapes and have the children copy them with their yarn.



Discuss the fact that the first figure is a simple closed curve. It begins and ends at the same point. The other 2 figures are not simple closed curves; 1 crosses itself and the other does not begin and end at the same point.

Have the children make other figures of both types. Discuss their figures to help them distinguish between the 2.

An infinite number of lines pass through a point. One way to illustrate this is with an overhead projector and a series of transparent overlays (Figure 8-7). Show a single line passing through a point on the first overlay (a). Show a second overlay that contains more lines passing through the point (b). With each successive overlay, show more lines passing through the point. Point out that there are also other lines passing through the point in

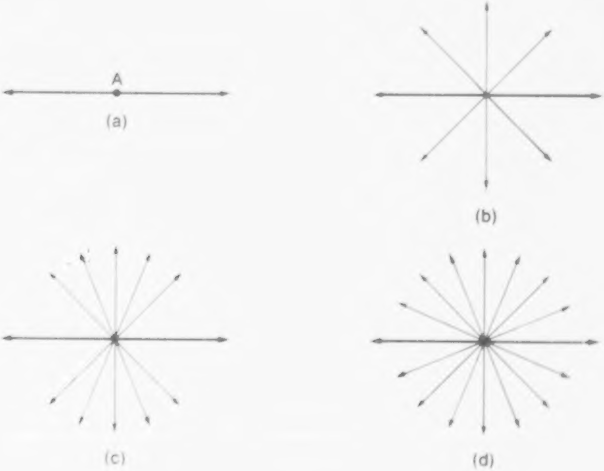


FIGURE 8-7

By using transparencies and overlays, one can show that many lines pass through a point.

directions other than those that are on the plane; these cannot be shown on the transparencies.

Perpendicular and parallel lines are commonly introduced to children in grades four through six. Your classroom provides models for both kinds of lines. The line segments formed where a wall joins the floor and the ceiling provide a model of parallel lines. Children can imagine that if the 2 lines were extended infinitely in both directions, they would never meet. The line formed by 2 walls joining at a corner and the line formed by the floor and 1 wall serve as models for perpendicular lines. Your children should be challenged to find other models of these 2 line relations in and out of the classroom.

LOGO EXPLORATIONS IN GEOMETRY

Logo provides students with opportunities to learn important mathematics concepts, including many from the field of geometry. Children's first work with Logo is usually with a part of the Logo language called Turtle Geometry. Turtle Geometry is named for a large, dome-shaped robot shaped like a turtle that followed computer commands as it moved around the floor in a laboratory at the Massachusetts Institute of Technology where Logo was created. The "turtle" is now a shape that moves about on the graphics screen of a computer running Logo. In many versions of Logo, the shape is a triangle, though some versions actually have an outline of a turtle move on the screen.

Children cause the turtle to move by typing commands on the keyboard. They usually learn the distance and direction commands first. FORWARD and BACK are distance commands that move the turtle forward or backward in a straight line. The direction commands RIGHT or LEFT turn the turtle. The size of lengths and angles are determined by numbers called inputs, typed with the commands. For example, if a child types

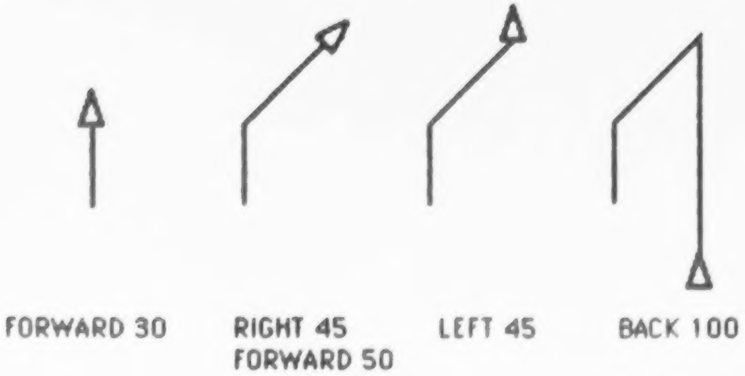
```
FORWARD 30
RIGHT 45
FORWARD 50
LEFT 45
BACK 100
```

the turtle goes forward 30 turtle steps, turns right 45 degrees, moves 50 steps forward, turns left 45 degrees, and finally goes back 100 steps leaving a drawing in its path (Figure 8-8).

The creators wanted Logo to be a learning environment in which mathematics becomes a natural language for describing events. Through Logo,

FIGURE 8-8

Beginning turtle moves in Logo



children learn to use numbers and geometry as a way to develop and record their ideas. Rather than have directed lessons on shapes and degrees, however, children learn about them by observing the path of the Logo turtle on the screen. Although children work with different shapes, they all use information about lengths, angles, and directions. By moving the turtle around the screen, children find that shapes are drawn when they type certain numbers with distance and direction commands. A square is often the first shape discovered through a guess-and-check approach using estimation. A child may work for an hour on a "square" corner trying RIGHT 110, 85, or 92 before finally getting the correct command—RIGHT 90—to work. After a square corner (Figure 8-9a) is made, making the rest of a square is easy.

```
FORWARD 40
RIGHT 90
FORWARD 40
RIGHT 90
```

The square is completed with 2 more repetitions of FORWARD 40 RIGHT 90. Later, children learn that the "invention" of a square can also be expressed in a REPEAT statement that eliminates extra typing and emphasizes the pattern of commands (see Figure 8-9b).

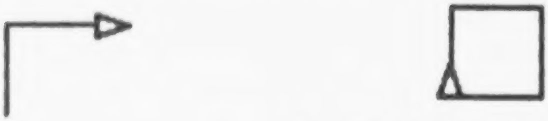
```
REPEAT 4 [FORWARD 40 RIGHT 90]
```

Other squares, as well as triangles, hexagons, octagons, and circles follow the first square. You may want to have your students keep a log or notebook of Logo shapes and commands as reference in order to organize their experiences.

As children become more sophisticated with their knowledge of Logo, projects become more complex. Children now need additional features of

FIGURE 8-9

A square corner becomes a square using a pattern of commands.



```
REPEAT 4 [ FORWARD 40 RIGHT 90 ]
```

```
FORWARD 40
RIGHT 90
FORWARD 40
RIGHT 90
```

the language to speed and ease their drawings. One of the most powerful aspects of Logo is the ability to create new commands by making a *procedure*. To make a procedure the commands that draw a shape or cause something else to happen are labeled. The command REPEAT 4 [FORWARD 30 RIGHT 90] makes a square. By typing TO SQUARE and the command, a procedure for making a square is made.

```
TO SQUARE
REPEAT 4 [FORWARD 30 RIGHT 90]
END
```

SQUARE is a short program that draws a square each time the command `SQUARE` is typed. The command could have been named BOX if used in a project requiring boxes or WINDOW if used in a house-drawing project.

Children typically create and save procedures for all of the basic shapes. With each new project, they encounter new problems that are solved by breaking down a complex idea into many smaller and easier ones. They may need special procedures for the drawing. The picture in Figure 8-10 involves simple shapes put together in different ways.

Some teachers use activity sheets or problem cards with shapes and designs to suggest projects. Textbook series now include Logo activities or have supplemental Logo workbooks. Many students create their own projects without additional materials. By doing so, however, they may overlook some of the concepts they are learning. Task cards suggest investigations that might not be discovered independently. Figure 8-11 shows a task card focusing on a mathematical idea, but it leaves room for students to go beyond the investigation. The teacher creates Logo procedures to draw SHAPE1, SHAPE2, and SHAPE3 for this task card. When you select or use additional materials, take care to avoid recipelike activities that give children only commands to type in. Recipes eliminate problem solving as a major benefit of Logo for students.

Although Logo should not be taught didactically, a teacher often helps

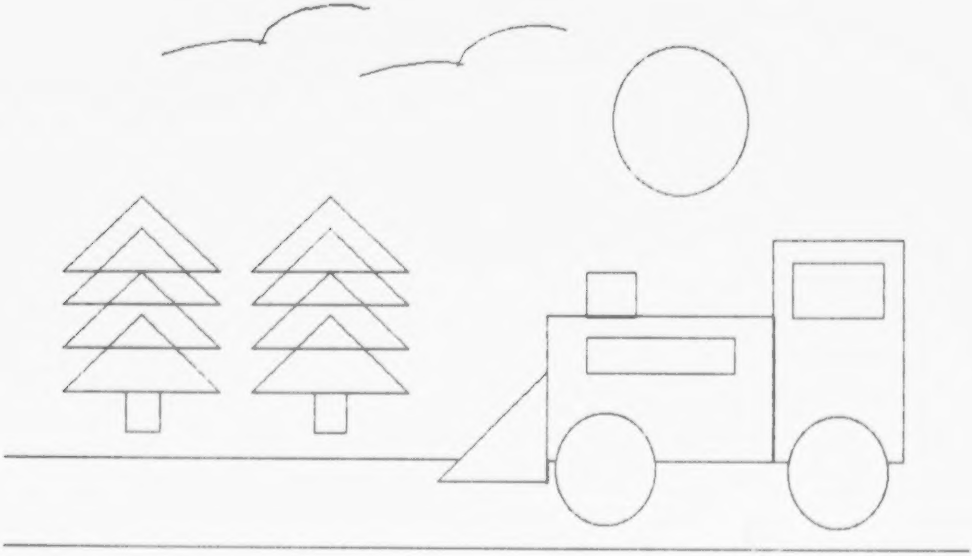
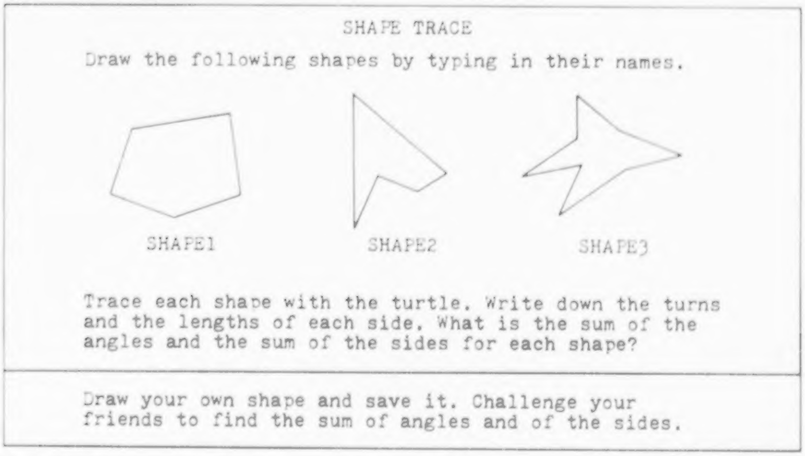


FIGURE 8-10

Example of a Logo project involving simple shapes combined into a picture

FIGURE 8-11

Task card for SHAPE TRACE



SHAPE TRACE

Draw the following shapes by typing in their names.

Trace each shape with the turtle. Write down the turns and the lengths of each side. What is the sum of the angles and the sum of the sides for each shape?

Draw your own shape and save it. Challenge your friends to find the sum of angles and of the sides.

students gain closure on the important concepts they have used. A conversation between a fifth-grade student and a teacher illustrates how the teacher intervenes to focus attention on an important idea:

> *Teacher:* You have lots of shapes in your Logo log. Have you noticed something about the angles in all of the shapes?
>
> *Student:* Each shape has a special angle. A square always is 90. In my triangles it is always 120, and in the hexagon it is always 60. I can make all the sides in each figure the same size, but the angles are different for each shape.
>
> *Teacher:* The angles make the difference. Have you ever tried a shape with 7 or 11 sides? What angles do you think they would need? You might try to make a chart with the shapes, number of sides, and angles.

The teacher suggests that the student account for all possibilities and make the chart as problem-solving strategies. After a week, the teacher follows up:

> *Teacher:* You have an 11-sided polygon and a 7-sided polygon in your log now. How did you figure out the angle?
>
> *Student:* When I made the chart of shapes, I saw that the hexagon had an angle of 60 and the octagon had an angle of 45, so I tried out angles between 45 and 60. Finally 51 almost worked. But I noticed something else. A hexagon had 6 angles of 60, and 6 × 60 is 360. An octagon had 8 angles of 45, and 8 × 45 is 360. I decided to try dividing 360 by 7. The answer was 51.4, which is just about what I got. Getting an 11-sided polygon was easy. I just divided 360 by 11 to get the angle. 360 is the same as the degrees in a circle, isn't it?
>
> *Teacher:* You have discovered the important Rule of 360.

The teacher posed a problem, which brought the student to a new understanding called the Total Turtle Trip Theorem. The sum of the angles the turtle makes when moving around a closed polygon is 360 degrees, or a multiple of 360. The teacher's questions helped the student progress from describing each shape to generalizing about all of the shapes. A teacher must be alert to the important ideas explored in Logo and opportunities to focus on them without dominating the explorations. Some students will need more structure to state the Role of 360, but the teacher should continue to probe for the idea.

Turtle investigations provide students with a background of ideas about concepts that are introduced formally later. In the task card activity shown in Figure 8-11, the turtle demonstrates the concept of *perimeter* by tracing around the shapes. Later the same activity can be used with the new term when perimeter is introduced in fourth or fifth grade.

Logo was designed as an exploratory environment for mathematics, or a mathematics microworld.[3] Students can use Logo to explore topics such as variables, common fractions, graphs, statistics, probability, area and perimeter, divisibility rules, functions, and algebra. Regardless of the mathematics topic, Logo provides opportunities for students in elementary school and beyond to confront and solve a variety of problems. A fundamental value of Logo is that children are actively involved with the learning process and are in control of the computer environment as they work.

You will find articles about Logo in the *Arithmetic Teacher*. Both the *Computing Teacher* and *Journal of Computing in Mathematics and Science Teaching* contain regular Logo columns. The *Logo Exchange* is a monthly publication with Logo activities in mathematics and other subjects.

SPACE FIGURES

We live in a three-dimensional world, and children's experiences inside and outside the home are with objects that have three dimensions. It is recommended that children work with three-dimensional models of the space figures in school.

Play Activities

Commercial materials such as Geo Blocks[4] and the attribute materials mentioned in Chapter 6, the colored wooden or plastic blocks commonly found in preschool and kindergarten classrooms, and boxes and cardboard tubes of all shapes and sizes should be available for children's play. Initial activities should give children freedom to construct whatever they like. They will design and create many imaginative structures; some will show a surprising amount of creativity. Many children have a natural inclination to make patterns and structures that balance or are symmetrical. Though you do not need to develop the notion of symmetry during early play activities, you can comment on the way 2 sides of a pattern look almost alike or are the same.

---

[3] Seymour Papert, *Mindstorms* (New York: Basic Books), 1981.

[4] A set of Geo Blocks contains more than 100 hardwood pieces cut into a wide variety of shapes having edges up to 4 inches long. In addition to the blocks, there are a *Teacher's Guide for Geo Blocks* and *Problem Cards for Geo Blocks*. All are produced by the Webster Division of the McGraw-Hill Book Company, Manchester Road, Manchester, MO 63011.

Another activity that interests children is making their own structures to take home. Give children a collection of various-sized boxes and tubes, some glue, and tempera paint, then allow them freedom to make a truck, railroad engine, steamship, or whatever strikes their fancies. Again call attention to the way 2 sides look alike (if they do) and any other interesting aspects of their creations.

Examination and Discussion of Solid Figures

During play activities children gradually learn some of the terminology for parts of space figures—corner, edge, square, triangle, circle. You should encourage them to use these words, but do not force the specialized names and definitions too soon. As children indicate their interest by asking for names of a figure's parts, organize them into small groups to examine and talk about their models. Work with each group until children are making the proper associations of terminology and parts. Later you can make problem cards to guide their work in a classroom learning center. Use plastic or tagboard models, if possible, because children can conceive of the figure itself and its inside and outside more easily with hollow and, especially, transparent models than with wooden or other solid figures.

ACTIVITY 8-13

Planes and Solids

Give each child or pair of children a plastic model of a cube, pyramid, prism, or cylinder.

Discuss the plane figures associated with the models—square, triangle, rectangle, and circle—and the edges and corners. Discuss the fact that each figure has an inside and an outside.

Count the shapes of a particular kind in a model. "How many squares are there in your model? Are there any other shapes in it? What are they? How many are there? Are they all the same size?"

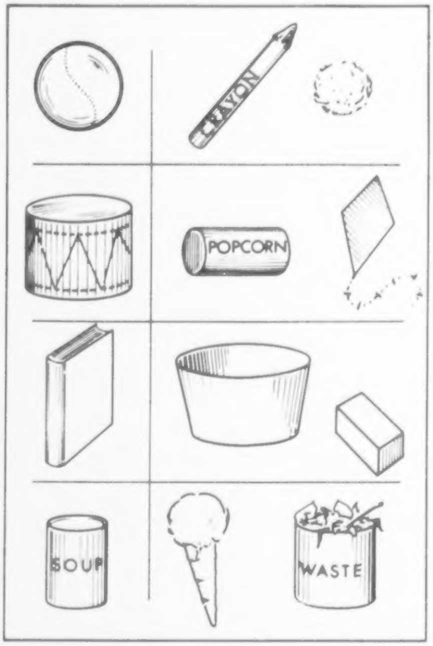
"How many edges are there in each model?"

"How many corners are there in each model?" (*Vertex* is the proper mathematical term for corner, but reserve its use for older children. Later, children will learn more precise definitions for each part of the space figures with which they work.)

"Which models roll easily? Which don't roll? Which slide when they are pushed? Which roll when placed 1 way but slide when placed the other way?"

FIGURE 8-12

Sample work page for shape recognition by primary-grade children



Other activities for young children include these:

> Have children take a geometry walk around the school grounds to identify plane and solid figures. Discuss whether each object is natural or human-made and whether it has utilitarian or aesthetic value, or both.[5]
>
> Have children complete work sheets matching or drawing lines between pictures of similar objects (Figure 8-12).
>
> Have children make a bulletin board or individual booklets with magazine pictures of objects that look like the space figures they have studied.

The following activity is appropriate for grades four through six. It helps children clear up confusion that often surrounds textbook pictures of space figures by having them draw sketches of different figures. Use both solid and transparent figures so children can distinguish between pictures with broken lines and those without.

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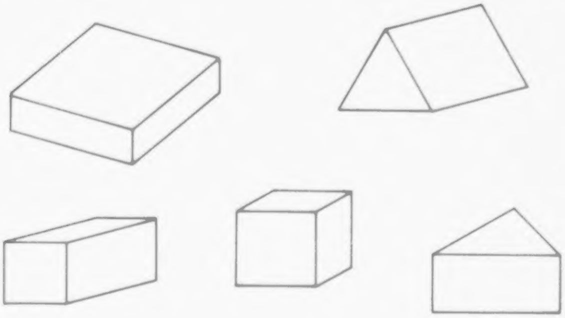
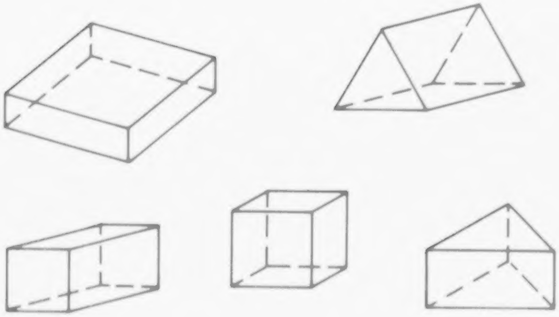
[5] See Glenn Nelson and Larry P. Leutzinger, "Let's Take a Geometry Walk," *Arithmetic Teacher*, No. 3 (November 1979), pp. 2–4

ACTIVITY 8-14

Solid Perspectives

Put 1 of these models in the center of a table or a group of desks.





Have children sit in groups of 5 or 6 around each table or group of desks. Tell the children that each one is to draw a pencil sketch of the figure as he/she sees it. If it is a solid model, the picture should not have broken lines to indicate positions of unseen edges; if it is a plastic model, broken lines should be used to show edges that are behind 1 or more of the faces.

After each sketch is completed, distribute them to the children so that none is given to the person who drew it. Direct the children to move about the room until each child believes he/she has found and is sitting in the chair in which the person who drew the sketch was sitting when the sketch was made. When 2 or more children claim the same chair, have them see if they can resolve the situation without your help.

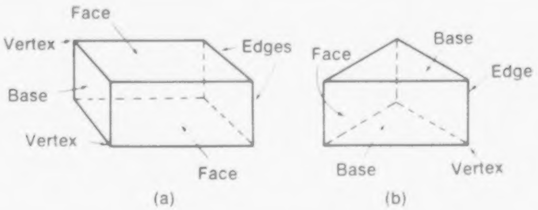


FIGURE 8-13

Identification of the parts of (a) a rectangular prism and (b) a triangular prism

Clear plastic models are ideal for older children's analysis of space figures such as prisms (Figure 8-13) and pyramids (Figure 8-14). They should learn that a right prism has 2 parallel bases that are congruent polygons and sides (faces) that are rectangles. If the bases are regular polygons, such as squares or equilateral triangles, the faces will be congruent rectangles. If the bases are not regular, such as a rectangle or an isosceles triangle, not all of the rectangular faces will be congruent. In a right prism, the faces are perpendicular to the bases so that they form right angles where they meet.

Pyramids have 1 base and triangular faces that meet at a point at the top, or apex. The pyramids studied in the elementary school usually have regular polygons as bases, although a pyramid may contain polygons that are irregular. Both prisms and pyramids are named according to the polygon that forms their bases. In Figure 8-13 there are (a) rectangular and (b) triangular prisms; and in Figure 8-14 there is a pentagonal pyramid. Polygons with any number of sides can form the bases of these figures.

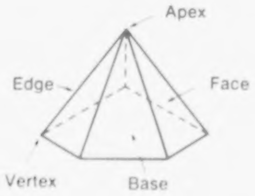


FIGURE 8-14

Identification of the parts of a pentagonal pyramid

During their study, children should learn the proper names for each figure and its parts. The parts named in Figures 8-13 and 8-14 are names that children should learn. They should also learn that these figures are polyhedrons. A polyhedron is a closed figure formed by the union of 4 or more polygons and their interiors.

The solid figures associated with the circle—sphere, cylinder, and cone—should also be studied (Figure 8-15).

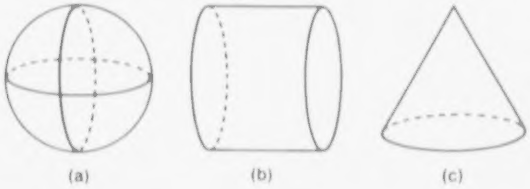
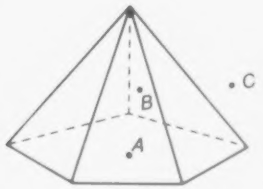


FIGURE 8-15

Common geometric solids associated with the circle

FIGURE 8-16

A polyhedron separates space into 3 distinct sets of points. Point A is on the pyramid itself, point B is inside the pyramid, and point C is outside the pyramid.



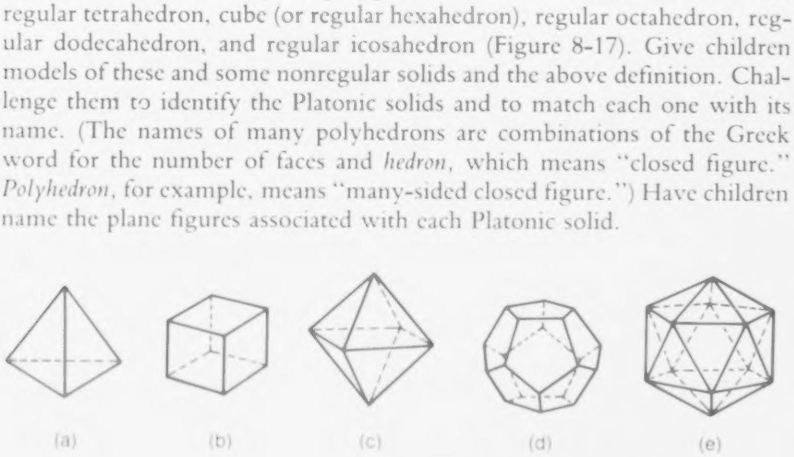
Commercial or teacher-made clear plastic models of these figures should be used rather than solid ones, if possible, so that children can visualize points on, inside, and outside each model (Figure 8-16). They can grasp the idea that polyhedrons or figures associated with the circle separate space into 3 distinct sets of points: the set of points that are a part of the figure itself; the set of points that occupy the interior of the figure; and the set of points found outside of the figure.

Here are other investigations with solid figures for older elementary-school children. Directions for each of these can be put on problem cards to make interesting learning center activities.

There are 5 figures called Platonic solids that are formed by the union of 4 or more congruent regular polygons and their interiors. These are the regular tetrahedron, cube (or regular hexahedron), regular octahedron, regular dodecahedron, and regular icosahedron (Figure 8-17). Give children models of these and some nonregular solids and the above definition. Challenge them to identify the Platonic solids and to match each one with its name. (The names of many polyhedrons are combinations of the Greek word for the number of faces and *hedron,* which means "closed figure." *Polyhedron,* for example, means "many-sided closed figure.") Have children name the plane figures associated with each Platonic solid.

FIGURE 8-17

The 5 Platonic solids: (a) regular tetrahedron, (b) cube (regular hexahedron), (c) regular octahedron, (d) regular dodecahedron, and (e) regular icosahedron.


(a) (b) (c) (d) (e)

Many different types of models should be available for children to study. Children can make some themselves. Plasticine clay or Play-Doh can be used for nonhollow figures. These models are especially good to have on hand when children investigate symmetry of solid figures. (Symmetry is considered later in this chapter.) Soda straws connected with pipe cleaners and glue make good skeleton models (Figure 8-18). Skeleton models can also be made by joining lengths of pipe cleaners. Tagboard figures can be made from printed patterns that are cut and assembled by the children. As

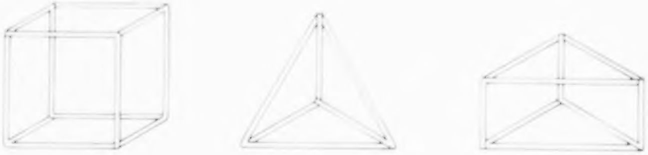


FIGURE 8-18

Soda-straw geometric solids

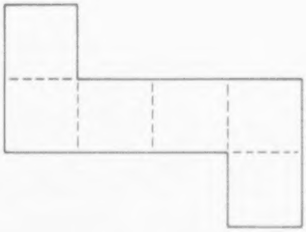
children make their own figures, they learn about edges, corners, the interiors of plane figures, and other parts of solid figures.

An interesting problem with tagboard models of solid figures is the study of nets. A net is made by cutting some of the edges of a figure so that the figure will lie flat or by drawing the outline of a figure and cutting it out. Two different nets for a cube are shown in Problem 8-2.

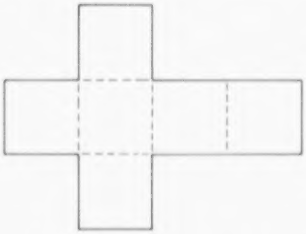
PROBLEM 8-2

Nets of Cubes

Show the children the first net and demonstrate how it can be folded to form a cube.



Show the second net and demonstrate how it is folded to form a cube.



Give each child several pieces of paper that have been covered with 2-inch squares. Let the children work alone or in pairs to create other nets for a cube. Tell them that there are more than 5 nets for a cube. (There are 7 different nets.) They can cut out their models to see if each one does form a cube. They may want to tape the edges of those that do.

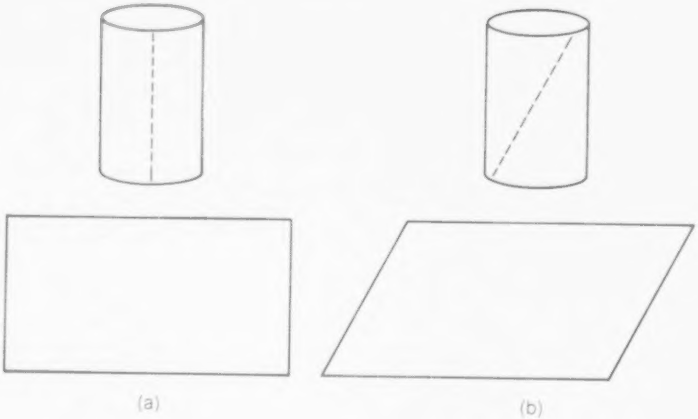


FIGURE 8-19

(a) A cylinder cut square with the edges gives a rectangle when it is flattened. (b) When the cylinder cut is oblique, flattening gives a parallelogram.

Nets for other figures—pyramids, the 5 Platonic solids, other polyhedrons, and models of spheres, cylinders, and cones—should be included in the children's investigations too. Children will find that a sphere cannot be flattened. They will also find that if the top and bottom of a cylinder are removed and the side is cut straight down from 1 edge to the other, a rectangle is formed when the figure is flattened, as in Figure 8-19 (a); but if the cut is an oblique one, the flattened figure is a parallelogram as in (b).

TRANSFORMATIONAL GEOMETRY

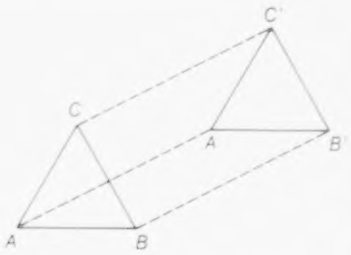
Transformational geometry is an area of study that will intrigue many students. It deals with 3 basic motions that can be applied to geometric and other figures: translations, or sliding motions; rotations, or turning motions; and flipping motions. Each motion is defined as a rigid motion, because it follows certain rules. Only when a motion adheres to these rules does a transformation occur. The figure obtained as a result of a transformation is congruent with the original figure and can be called a *congruency transformation*. The figure that comes from a transformation is also an image of the original figure.

Children's first experiences should be exploratory, with investigations that refine their understanding of concepts of congruency, symmetry, parallelism, and the nature of plane figures. Slide motions can be visualized with a piece of tissue paper on which figures are traced and then moved; flip motions can be visualized with wire models of plane figures.

A slide motion occurs when a geometric figure moves along a plane so that the distance between all corresponding points on the 2 figures remains

FIGURE 8-20

The slide of triangle ABC transforms it into triangle A'B'C'.



the same. A slide motion involving a triangle is illustrated in Figure 8-20. The original triangle *ABC* has been transformed to triangle *A*'B'C' by the slide motion represented by the dotted lines.

Flip motions create a reflection of the original figure. During their examination of these motions, children will extend their understanding of symmetry and of regular plane figures. A flip action may take place in relation to a point or a line. In Figure 8-21(a), parallelogram *A*'B'C'D' is the reflection of parallelogram *ABCD*. The line between the 2 parallelograms is equidistant from each and is the line of symmetry, or *flip axis*. In (b) the reflection is in respect to a point at the center of the figure. Children should be guided to observe that symmetry of the type considered here is found frequently in nature. The 2 halves of a butterfly have line symmetry; the parts of many flowers have point symmetry. Children should use mirrors to determine symmetry of figures. When a mirror is placed along the line in (a) and between halves of the drawing in (b), it will reflect figures that duplicate the originals.

A flip action will help children develop their understanding of the meaning of perpendicular lines. Two lines are perpendicular to each other when the reflection of 1 remains unchanged after it has been flipped along the axis of the other line (Figure 8-22). Children can demonstrate this with pieces of wire bent in the shape of angles (Figure 8-23). The wire should be placed on a piece of paper so a tracing can be made of it. Then it should be flipped along 1 axis and another tracing made. When the line that is not the axis continues as a part of the original nonaxis line, the 2 lines are perpendicular (a). When the line does not continue as a part of the same line, they are not perpendicular (b).

FIGURE 8-21

A flip may take place along the vertical line drawn in (a) or along a vertical or horizontal line in (b).

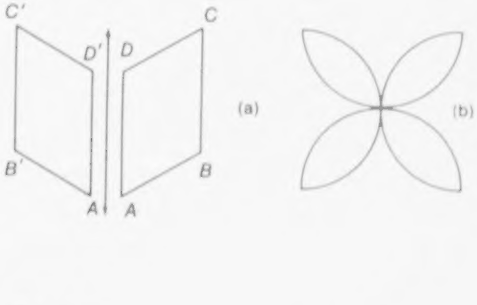


FIGURE 8-22

Two lines are perpendicular if the line m is invariant when a flip is made on the line l.

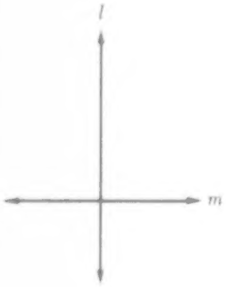
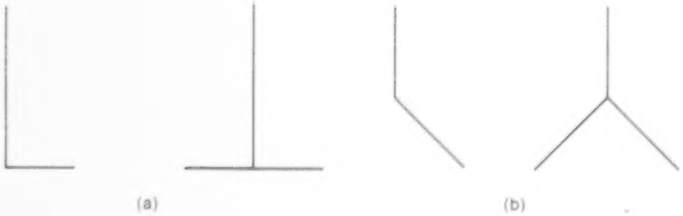


FIGURE 8-23

Pieces of wire can be used to determine whether 2 lines are perpendicular. In (a) the tracing indicates that the lines are perpendicular; in (b) they are not.



CONGRUENCE AND SIMILARITY

Two line segments are congruent when all points on one segment coincide exactly with all points on the other. Similarly, 2 plane figures or 2 space figures are congruent when all points of 1 figure coincide exactly with all points on the other. This is the concept on which standardized measurement, with its units for linear, area, and volume measures, is based.

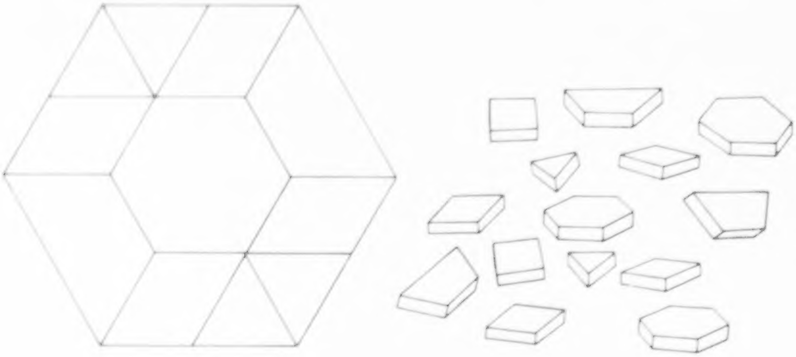
Two figures are similar when they have the same shape and all of their corresponding parts are proportional. Thus all squares, except those that are congruent, are similar, whereas not all rectangles are. Other similar shapes are equilateral triangles, regular pentagons, hexagons, and circles, except when 2 like shapes are congruent. Two drawings of a room's floor plan are also similar if they are drawn to different scales.

Congruence

You can use several types of activities to introduce congruence or extend children's understanding of it. Problem cards can be prepared for many of them.

FIGURE 8-24

Pattern block pieces show a model to be copied



a. Pattern Blocks Give pairs of children a set of pattern blocks (Figure 8-24). Have 1 child choose and put together a pattern for the other child to copy.

b. Attribute Blocks Have children separate a set of attribute blocks into groups of congruent blocks.

c. Geoboards You can set up a geoboard with line segments, angles, and open and closed figures for children to copy. Leave enough space next to each figure for a child's copy of it, or give children individual boards.

d. Tracings Cut out line-drawing figures of animals, automobiles, other objects, and plane figures and glue them on cards, or draw simple line figures on cards for children to trace. Use either lightweight paper or clear plastic for the tracings. Figure 8-25 shows a triangle *ABC* and its tracing *MNO*. Tracings can also be used to show that 2 figures are not congruent. In Figure 8-26, the tracing of *XYZ*, represented by the broken lines on *DEF*, shows that the 2 angles are not congruent.

FIGURE 8-25

A triangle ABC *and its tracing* MNO

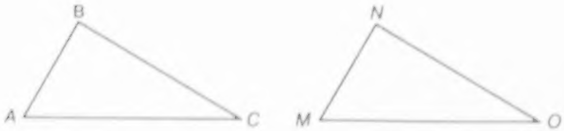
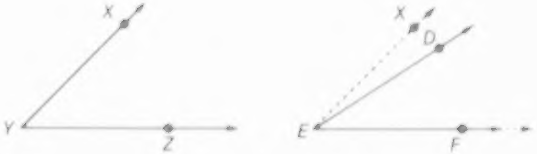


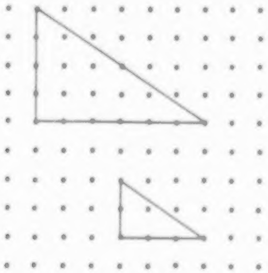
FIGURE 8-26

The tracing of ∠XYZ *placed over* ∠DEF *shows that the 2 angles are not congruent.*

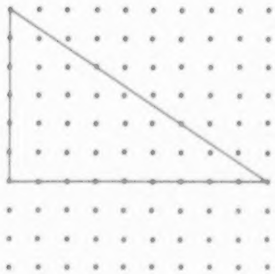


ACTIVITY 8-15 Similarity

- Give each child a geoboard and several rubber bands.
- Show a right triangle on your geoboard. Instruct the children to make one that is like yours but with sides only half as long.



- Discuss the differences between your triangle and those of the children. Make certain that the children realize that each side of their triangle is half as long as the corresponding side on yours.
- Now have the children make a triangle with sides 1½ times as long as yours.



- Compare this triangle with the other 2.
- Tell the children that 2 figures that look alike but are not the same size are *similar* figures.
- Repeat with other figures: squares, rectangles, parallelograms, and so on.

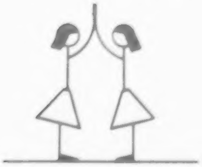
SYMMETRY

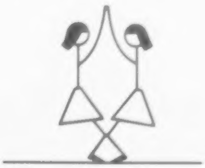
Symmetry is found in both natural and human-made objects. Many geometric figures are symmetrical. Investigations in this area of geometry refine children's understanding of concepts such as congruence and the nature of plane figures. They also help children develop an appreciation of some applications of symmetry in art. The activities that follow will acquaint you with ways to introduce the concept of symmetry and broaden children's understanding of it.

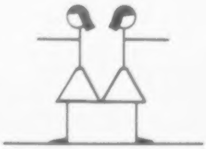
ACTIVITY 8-16 Symmetry

Select 2 children who are the same sex and about the same size.

Tell them to stand on a table at the front of the room, face each other, and join hands. Give them these instructions: "Raise your hands high above your heads." "Keep your hands joined above your heads as you bend your knees until they touch." "Now let go of your hands and turn around so your backs are touching. Put your hands straight out in front of you, then push against each other and slowly move your feet forward so you look as though you are going to sit down."







Have the pair of children model other similar types of motions.

Have children from the class tell you what they noticed each time the positions changed. These responses might be expected: "Each one looked like the other after they moved." "They always did the same thing." "It looked like one was in front of a mirror and we saw her image in it."

ACTIVITY 8-17 Symmetrical Blobs

Give each child a piece of white art paper and access to several pots of colored tempera paint.

Instruct the children to fold their papers in half. (The fold can be made either the long or the short way.)

Have the children open their papers, put blobs of different colored paint on 1 side, and fold the clean side over the paint. Smooth the 2 halves so the paint is spread evenly on both halves.

When the paint is dry, mount the papers on a bulletin board.

Discuss the fact that 1 side is the mirror image of the other; the 2 sides are symmetrical.

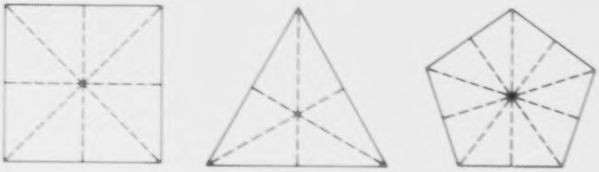
Once the concept of symmetry is understood, you can introduce problems dealing with lines and points of symmetry. Paper-folding and posting-hole problems are interesting to children in grades four through six.

PROBLEM 8-3

Lines of Symmetry

Give children paper models of a square, rectangle, equilateral triangle, isosceles triangle, regular pentagon, and other polygons.

Have children work in groups of two or three to determine the number of different ways each figure can be folded so that 1 half exactly covers the other half. (The illustrations show the folds for a square, an equilateral triangle, and a regular pentagon.) Tell the children that each fold line is called a line of symmetry.



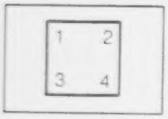
Tell the children to make a chart on which to record their data.

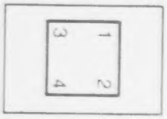
Challenge the children to determine if there is any relationship between the number of sides and the number of lines of symmetry in a regular polygon. Does the relationship exist for a rectangle or an isosceles triangle? What about the other figures? Are there any figures that have no lines of symmetry? What can be said about their shapes?

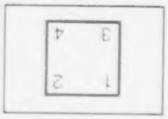
PROBLEM 8-4

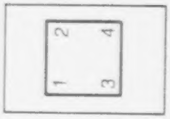
Rotations

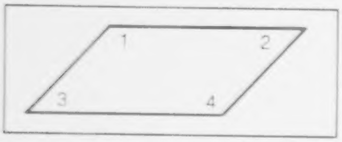
Give each group of 2 or 3 children a set of posting holes and their inserts. Have them number the corners of each insert as shown in the first picture of the square and the parallelogram on the next page.

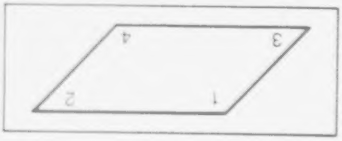












Beginning in the starting position, how many times can each insert be fitted into its posting hole during 1 rotation?

Have each group of children make a chart on which to record their data.

Ask the children to determine if there is any relationship between the data they gathered in this problem and the data gathered in Problem 8-3.

PROBLEM 8-5 Rotational Symmetry

For this problem, children will need to bring in lidded boxes of various shapes and sizes. (The more odd-shaped the boxes are the more interesting the problem will be.)

Challenge the children to determine if any of the boxes have lids that can be placed on the box only one way. Such a box does not have rotational symmetry. A square box has fourfold rotational symmetry because its lid will fit 4 different ways. How many different ways can the lid be placed on each of the other boxes? What number of placements can be made with a circular lid on its box? With an oval lid on its box?

Have the children make a chart to record their data.

Have the children determine which boxes have the same number of placements for each rotation as there are number of sides, and which boxes have fewer placements for each rotation than there are number of sides.

Have the children compare the chart from this problem with the charts for Problems 8-3 and 8-4.

By the time children have finished their work with these 3 problems, they should come to the conclusion that a regular plane figure has as many lines of symmetry as there are number of sides, while a figure that is not regular has fewer lines of symmetry than it has sides or no lines at all.

FIGURE 8-27

Block letters for symmetry activities

A B C D E F G H I J K L M N

O P Q R S T U V W X Y Z

Other symmetry activities can be organized around these materials:

Children can collect and prepare pictures of symmetrical natural and human-made objects for a bulletin board.

A line drawn from the upper left to the lower right in the addition table (Figure 9-6) and the multiplication table (Activity 10-6) separates each of the tables into symmetrical halves. It is the symmetry of these tables that makes it possible to show that the commutative property applies to each operation.

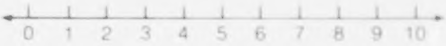
Block letters can be investigated for line and rotational symmetry (Figure 8-27). Consider all letters and note that certain ones are symmetrical along a vertical line (A and T are examples); along a horizontal line (B and C are examples); or both ways (I and O are examples). The latter 2 letters also have rotational symmetry.

COORDINATE GEOMETRY

René Descartes (1596–1650) is credited with the development of coordinate geometry. Elementary-school children can be introduced to the coordinate plane through activities that are both practical and fun. A prerequisite skill is that children be familiar with number lines. You can introduce the coordinate plane to children who have no knowledge of negative numbers, although knowledge of them is necessary to work with the complete coordinate grid. The first activity is designed to develop children's understanding of the plane. It begins with the whole-number line, which it uses as the basis for constructing the coordinate system. Other activities suggest ways children's understanding and use of the system can be extended.

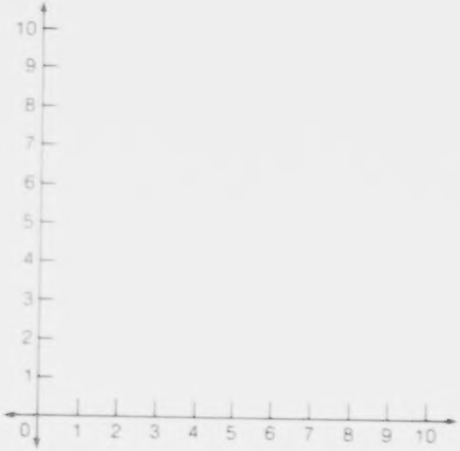
ACTIVITY 8-18 The Coordinate Plane

Make a horizontal drawing of a whole-number line on the chalkboard. Mark the points 0-10.

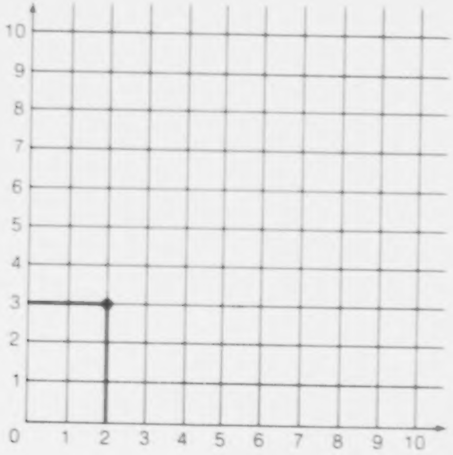


Review the way counting is done on a number line.

Draw a vertical number line that has the same origin as the first line, and name its points.



- Have the children demonstrate how counting is done on this line.
- Tell the children that the horizontal line is the first axis and the vertical line is the second axis.
- Introduce the concept of ordered pairs by drawing a star, square, circle, or other simple figure on one of the points of the plane (say 2,3). Point out to the children that in the pair, called the coordinates of the point, the first number indicates the number of units to count on the first, or



horizontal, axis; and the second number indicates the units to count on the second axis. Mark a dark line on your coordinate plane to show the sequence of steps for locating a point.

Locate other points on the plane and have children identify each one's coordinates.

Name coordinates and have individuals come to the board to show the order of steps for locating each one, and then indicate each one's position.

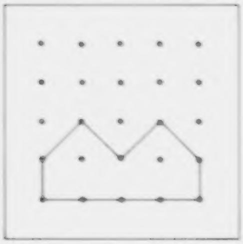
Variation: If your children have knowledge of integers, begin work with the coordinate plane by using all 4 quadrants during the introductory lesson.

Each of the following suggestions is representative of activities you can design or find in journals such as *Arithmetic Teacher* and in teacher's and resource manuals and children's workbooks.

Use a geoboard for a coordinate plane by locating the intersection of the horizontal and vertical axes in the lower left corner. Children can demonstrate understanding by putting small washers on points for coordinates you name. They can put rubber bands on the board to represent simple pictures. Figure 8-28 shows a duplex that was made using a rubber band to connect coordinates.

FIGURE 8-28

Duplex constructed using geoboard as the coordinate plane



Children can locate points on dittoed grids and connect them with line segments to make pictures. The evergreen tree in Figure 8-29 is made by connecting the coordinates named next to it.

Exploration of positive and negative numbers and the coordinate plane on a computer is available through the *Bumble Games*.[b] In 6 different games children have opportunities to deal with these concepts at different levels of complexity.

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[b] The Learning Company, 6493 Kaiser Drive, Fremont, CA 94555.

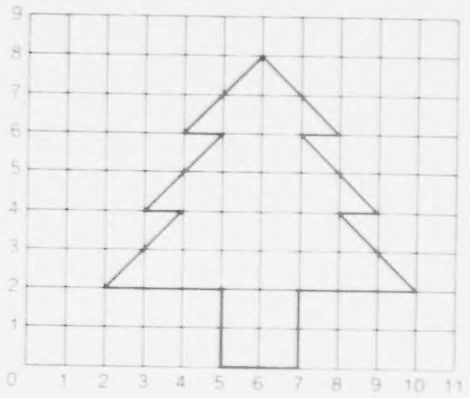


FIGURE 8-29

Evergreen tree constructed on a coordinate plane, using the coordinates: (5,0) (7,0) (7,2) (10,2) (8,4) (9,4) (7,6) (8,6) (6,8) (4,6) (5,6) (3,4) (4,4) (2,2) (5,2) (5,0)

THE VALUE OF GEOMETRY FOR STUDENTS WITH SPECIAL NEEDS

The geometry topics treated in this chapter have special value for many children with special needs. Topological activities will help certain learning-handicapped children, such as those with figure-ground and spatial problems, develop readiness for both the mathematics and reading and writing activities that lie ahead of them in school. As these children recreate bead patterns from problem cards and make their own and deal with enclosure on a plane, they develop an understanding of open, closed, inside, outside, and around, which are skills essential for learning the order of numbers and the configuration of numerals and letters. Frequent activities dealing with proximity help these children learn to identify and use relations such as far, near, close to, next to, and so on.

The very nature of the activities associated with a study of geometry is beneficial to many physically handicapped children. The manipulative materials that are used, such as shapes, posting holes, and models of geometric figures, not only develop understandings of geometric concepts but can also help students develop both gross and fine motor skills. While not all of the commercial materials are suitable for children with motor-skill deficits, larger-sized models can be made. A large-sized set of posting holes, a geoboard with large, round-headed brads spaced 1½ to 2 inches apart, and dot paper with 1-inch spaces between dots are easily made. Alterations in the way some children are asked to respond may also be necessary. Some children may be unable to copy a geoboard shape on dot paper; a visual inspection of their work on the geoboard may be necessary. Some children may need to give oral rather than written responses for some activities. In these cases you should alter the manner of presentation, the amount of time devoted to an activity, and children's manners of responding, not the mathematics content with which children deal.

Summary

Geometry is a part of contemporary mathematics in the elementary school. There is substantial agreement that children profit from work with it. Piaget's research indicates that children first have a topological view of their world rather than a Euclidean view. Topologically oriented activities are recommended for preoperational-level children. Four topological concepts are discussed: proximity, separation, order, and enclosure. P. van Heile has determined that individuals pass through 5 levels of knowledge before they develop an understanding of Euclidean geometry. The first 3 of his 5 levels are of concern to elementary-school teachers. They deal with recognition of simple geometric figures, recognition of each figure's major characteristics, and classification of figures according to their attributes.

All work should be informal; investigations with models of figures predominate. Children learn about plane and space figures, as well as those associated with the line. Logo offers many opportunities for problem solving as children explore geometric figures on a computer. Children become acquainted with concepts of similarity, congruence, symmetry, and transformations.

Activities dealing with topological concepts help children with special needs acquire some of the skills and understandings they need to work with other areas of mathematics and with numerals, letters, and words. Both children with learning handicaps and those with physical handicaps benefit from geometry activities when materials and procedures are adapted to their special needs.

Study Questions and Activities

1. Many activities in geometry are well suited for adaptation to problem cards in order to guide children's work in laboratories or learning centers. Select a geometry topic and design a set of cards suitable for children. For example, you might design a set to guide work about the three types of triangles and their altitudes shown in Figure 8-5. Another set of cards could deal with transformational geometry. The articles listed in the For Further Reading section will be helpful in choosing other topics.
2. Glenda Lappan and Mary Jean Winter (see For Further Reading section) discuss a sequence of activities for improving children's spatial visualization skills. Use their article as a basis for designing a spatial visualization unit for fourth, fifth, or sixth graders.

3. Glenn Nelson and Larry P. Leutzinger (see For Further Reading section) recommend that children learn about geometry by taking a "geometry walk." After reading their article, walk around an elementary school or your university classroom building with the aim of planning a geometry walk. The article includes suggestions about the types of questions you can ask during the walk. Write questions and possible responses in the same manner as those in the article.
4. Become a friend of the turtle by spending some time working with Logo. Try making squares, triangles, hexagons, octagons, and circles. What geometric ideas did you discover or rediscover for yourself? Read Bill Craig's article (see For Further Reading section) for some ideas about working with students with Logo.
5. The article by Barbara Renshaw (see For Further Reading section) suggests using company trademarks for a symmetry lesson. Make a collection of pictures, including trademarks, that illustrate symmetrical figures for a classroom bulletin board. Include examples of both plane and space figures having point symmetry and line symmetry. Design a board with which children must interact in some way.
6. Several articles in the For Further Reading section suggest activities for tiling and tesselations, which are very popular activities with students. Use the ideas from van de Walle and Thompson to create geometric designs with paper. Which ideas from this chapter can you demonstrate with cut-and-paste designs? If you are working with children, plan a lesson with cut-and-paste.

For Further Reading

Arithmetic Teacher, XXXVI, No. 6 (February 1979). Five articles in this issue deal with geometry. Activities and materials for all grade levels are included.

Battista, Michael T. "Distortions: An Activity for Practice and Exploration," *Arithmetic Teacher*, XXIX, No. 5 (January 1982), pp. 34–36. Morris the cat serves as the basis for a study of the plane coordinate system. He is first drawn on a normal grid. Then a variety of grids that distort the cat's appearance focus attention on different grid forms and the transformations that occur on them.

Bright, George W. "Using Tables to Solve Some Geometry Problems," *Arithmetic Teacher*, XXV, No. 8 (May 1978), pp. 39–43. Describes and illustrates 5 problems dealing with shapes contained in square and triangular grids. Each one uses a table to organize data and reveal patterns that can be extended beyond the materials themselves.

Burger, William F. "Graph Paper Geometry," *Mathematics for the Middle Grades (5–9)*, 1982 Yearbook, National Council of Teachers of Mathematics (Reston, Va., The Council), pp. 102–117. Activities with the coordinate plane introduce coordinates, patterns of shapes and tessellations, and symmetry and motions. Coordinates for 9 different activities are given.

Butzow, John W. "Y is for Yacht Race: A Game of Angles," *Arithmetic Teacher*, XXXIII, No. 5 (January 1986), pp. 44–48. Materials and activities for upper elementary students to explore angles by plotting the path of a sailboat.

Cangelosi, James S. "A 'Fair' Way to Discover Circles," *Arithmetic Teacher*, XXXIII, No. 3 (November 1985), pp. 11–13. Attributes of circles and polygons are explored.

Craig, Bill. "Polygons, Stars, Circles, and Logo," *Arithmetic Teacher*, XXXIII, No. 9 (May 1986), pp. 6–11. Explores simple shapes with Logo and the mathematical ideas that can be learned from them.

Holcomb, Jean. "Using Geoboards in the Primary Grades," *Arithmetic Teacher*, XXVII, No. 8 (April 1980), pp. 22–25. Kindergarten and first-grade children use geoboards to make designs and patterns, explore 3- and 4-sided figures, practice making block letters, and learn about open and closed figures. Examples of activities are explained and illustrated.

Horak, Virginia M., and Willis J. Horak. "Using Geometry Tiles As a Manipulative for Developing Basic Concepts," *Arithmetic Teacher*, XXX, No. 7 (March 1983), pp. 8–15. Geometry tiles are squares cut from heavy cardboard or flexible floor tiles. Directions for making tiles and activities for using them are explained.

Jensen, Rosalie, and David R. O'Neil. "Informal Geometry Through Geometric Blocks," *Arithmetic Teacher*, XXIX, No. 9 (May 1982), pp. 4–8. A set of teacher-made blocks (illustrated in the article) are used to develop geometric concepts and skills appropriate for kindergarten through grade three. Shape-comparison, classifying and naming, tracing, and measuring activities and discussions of properties are described.

Lappan, Glenda, and Mary Jean Winter, "Spatial Visualization," *Mathematics for the Middle Grades (5–9)*, 1982 Yearbook, National Council of Teachers of Mathematics (Reston, Va., The Council), pp. 118–129. Children are introduced to "buildings" made of blocks placed on mats. Perspective drawings are used to help visualize each building. Activities for drawing plans, constructing buildings from plans, measuring and designing buildings, and making three-dimensional drawings are included.

Lindquist, Mary Montgomery, ed. *Learning and Teaching Geometry*. Reston, Va.: National Council of Teachers of Mathematics, 1987. Twenty chapters discuss the geometry curriculum and the teaching of geometry.

Logo Exchange, International Council for Computers in Education, Department of Computer and Information Science. University of Oregon, Eugene, OR 97403. A magazine devoted to ideas and activities for Logo.

Martin, Kathleen, and Donna Bearden, *Mathematics and Logo: A Turtle Trip through Geometry*. Englewood Cliffs, N.J.: Prentice-Hall, 1985. This teacher's guide provides many activities for exploration of mathematics using Logo in the upper elementary grades.

Nelson, Glenn, and Larry P. Leutzinger. "Let's Take a Geometry Walk," *Arithmetic Teacher*, XXVII, No. 3 (November 1979), pp. 2–4. A geometry walk helps children learn the relevance of geometry to their lives. Suggestions of things to view and questions to ask during a walk are presented.

Olsen, Alton T. *Mathematics Through Paper Folding*. Reston, Va: National Council of

Teachers of Mathematics, 1975. Activities that emphasize geometric relationships. Especially good for older or gifted students.

Poggi, Jeanlee M., "An Invitation to Topology," *Arithmetic Teacher*, XXXIII, No. 4 (December 1985), pp. 8–11. Presents activities for exploration of topology.

Pohl, Victoria. *How to Enrich Geometry Using String Designs*. Reston, Va: National Council of Teachers of Mathematics, 1986. Although designed for grades 6–10, the elementary teacher will find many ideas for exploration of geometry.

Renshaw, Barbara. "Symmetry the Trademark Way," *Arithmetic Teacher*, XXXIV, No. 1 (September 1986), pp. 6–12. Finds examples of symmetry in many commercial trademarks.

Sawada, Daiyo. "Symmetry and Tessellations from Rotation Transformations on Transparencies," *Arithmetic Teacher*, XXXIII, No. 4 (December 1985), pp. 12–13. Tesselation is a process of organizing tiles systematically to form patterns. This article provides an excursion into tesselations.

Smith, Robert F. "Coordinate Geometry for Third Graders," *Arithmetic Teacher*, XXXIII, No. 8 (April 1986), pp. 6–11. Presents many activities that develop coordinate geometry with geoboards, dot paper, and maps.

Terc, Michael. "Coordinate Geometry—Art and Mathematics," *Arithmetic Teacher*, XXXIII, No. 2 (October 1985), pp. 22–24. Changing the scale on a drawing is easy with coordinate geometry.

Thornton, Carol A. "Geometry: Perceptual-Motor Help for Many Handicapped Learners," *Arithmetic Teacher*, XXVII, No. 2 (October 1979), pp. 24–26. The author believes that the benefits of geometry activities for handicapped learners are too great to be ignored. She indicates some activities for enriching children's perceptual awareness.

Tipps, Steve, and Glen Bull. *Beginning with Logo*. Englewood Cliffs, N.J.: Prentice-Hall, 1987. A guided exploration from the beginning of Logo concepts and commands.

Van de Walle, John, and Charles S. Thompson. "Concepts, Art, and Fun from Simple Tiling Patterns," *Arithmetic Teacher*, XXVIII, No. 3 (November 1980), pp. 4–8. Tessellations are repeating patterns formed with 1 or more plane geometric shapes. The shapes in this article are 2 different rectangles, a triangle, and a parallelogram. Basic patterns and the educational payoffs from work with them are described.

———. "Cut and Paste for Geometric Thinking," *Arithmetic Teacher*, XXXII, No. 1 (September 1984), pp. 8–13. Shows how paper designs with squares and triangles develop geometric concepts.

———. "A Triangle Treasury," *Arithmetic Teacher*, XXVIII, No. 6 (February 1981), pp. 6–11. Games and activities that use a set of triangle pieces (described in the article) give children experiences in comparing likenesses and differences, studying and measuring angles, making quadrilaterals from triangles, measuring perimeters, working with symmetry, generating solids, and making tessellations.

Walter, Marion. "Frame Geometry: An Example in Posing and Solving Problems," *Arithmetic Teacher*, XXVIII, No. 2 (October 1980), pp. 16–18. A set of 3-centimeter, 5-centimeter, and 8-centimeter sticks, representing materials for picture frames, is used to pose a problem about the number of different frames that can be constructed with them. The article outlines the paths different children might take as they work at solving the problem.

Watt, Dan. *Learning with Logo.* New York: BYTE/McGraw-Hill, 1983. Activity book with many Logo ideas.

Wenninger, Magnus J. *Polyhedron Models for the Classroom.* Washington, D.C.: National Council of Teachers of Mathematics, 1966. Presents illustrated directions for making polyhedron models. Includes many line drawings and photographs.

Young, Jerry L. "Improving Spatial Abilities with Geometric Activities," *Arithmetic Teacher*, XXX, No. 1 (September 1982), pp. 38–43. Suggests ways to enhance children's visualization of space by associating objects with their pictures or drawings, working with tangrams, seeing a shape as part of a larger figure, and using perspective and hidden lines.

Zurstadt, Betty K. "Tessellations and the Art of M. C. Escher," *Arithmetic Teacher*, XXXI, No. 5 (January 1984), pp. 54–55. Geometry and art are combined in tessellation projects.

Chapter

9

Teaching Addition and Subtraction

In Chapter 9, you will read about

1 the importance of real-life situations in developing understanding of addition and subtraction

2 four different real-life situations that are expressed in subtraction

3 developing readiness for addition and subtraction with discrete objects and a number line

4 procedures to introduce addition and subtraction facts and number sentences for them

5 four strategies for learning basic facts

6 the importance of appropriate practice for immediate recall of facts and practice activities including computer programs

7 using the commutative, associative, and identity properties of addition to aid in learning the facts

8 higher-decade addition and materials designed to help children understand and perform it accurately

9 using beansticks, chips, a place-value pocket chart, and an abacus to represent addition and subtraction of two-digit numbers with and without regrouping

10 two algorithms for computing subtraction

11 the problem children have in regrouping when zero is involved, and a strategy for helping them understand

12 common errors that children experience with addition and subtraction, and how recognition of error patterns helps a teacher adjust instruction

Chapter 1 contains steps for subtraction with whole numbers and a discussion of the importance of establishing and following a sequence of steps for each topic. This chapter discusses ways to present addition and subtraction to children. You should refer to the subtraction sequence in Chapter 1 and the one for addition that follows (Figure 9-1) as you read the chapter to see how the procedures and materials for learning about these operations fit into the sequence for each one.

A considerable amount of the time devoted to mathematics in the elementary school is spent on the operations of addition and subtraction and the development of understanding of an algorithm for computing each operation. Addition is described as the operation used to assign a sum to an ordered pair of numbers, which are called addends. Subtraction is described as the inverse of addition and is defined as the operation used to find a missing addend when a sum and the other addend are known. These definitions give us useful ways of summarizing addition and subtraction at the abstract level. However, they are not useful definitions for children, who need concrete ways of visualizing the 2 operations.

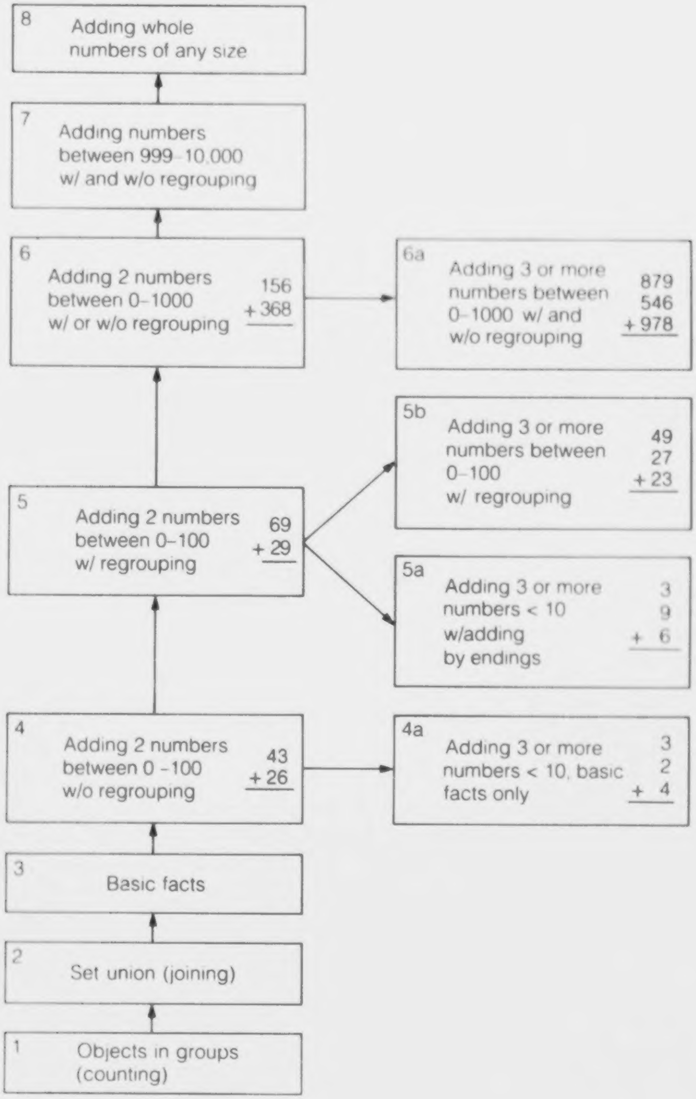
Numbers that are added are called *addends*, and their answer is a *sum*. When subtraction is defined in terms of addition, the numbers are a sum, a known addend, and a missing addend. There are special names for each of the numbers when they are used in subtraction. The minuend is the same as the sum, while the subtrahend is the known addend; and the *difference*, or remainder, is the missing addend.

BASIC FACTS

A basic addition fact is an ordered pair of whole number addends, each smaller than 10, and their sum. Altogether there are 100 addition facts that include all of the possible combinations using pairs of numbers less than 10 and their sums. Every addition fact has a subtraction fact that is its inverse, that is, one that reverses the action done by adding, so there are also 100 subtraction facts. Ultimately each child who is capable of doing so must learn each of the facts for rapid recall. Otherwise the child will be handicapped when working with an algorithm for each of the operations.

FIGURE 9-1

A series of steps for learning to add whole numbers


8 Adding whole numbers of any size
7 Adding numbers between 999–10,000 w/ and w/o regrouping
6 Adding 2 numbers between 0–1000 w/ or w/o regrouping 156 + 368
6a Adding 3 or more numbers between 0–1000 w/ and w/o regrouping 879 546 + 978
5b Adding 3 or more numbers between 0–100 w/ regrouping 49 27 + 23
5 Adding 2 numbers between 0–100 w/ regrouping 69 + 29
5a Adding 3 or more numbers < 10 w/adding by endings 3 9 + 6
4 Adding 2 numbers between 0–100 w/o regrouping 43 + 26
4a Adding 3 or more numbers < 10, basic facts only 3 2 + 4
3 Basic facts
2 Set union (joining)
1 Objects in groups (counting)

ADDITION AND SUBTRACTION SITUATIONS

There is only 1 type of real-world situation that gives rise to addition: the joining of 2 or more groups of objects. Subtraction, however, arises from several kinds of situations. Children should have experiences with these types:

> Subtraction is used when part of a set is removed. This is illustrated by the following situation: "A merchant began a sale with 396 boxes of greeting cards. During the first day, she sold 218 boxes. How many boxes did she have left?" This situation is a take-away type and is the one most frequently encountered by children.
>
> Subtraction is used to compare the sizes of 2 sets. In this situation, subtraction is used to find out how much larger or smaller 1 set is than another. "There are 2 schools in town. One school has 684 students, and the other has 478 students. The second school is how much smaller than the first?" (Or "The first school is how much larger than the second?")
>
> Subtraction is used to determine the size of a set to be united with another set of known size to make a third set of known size. "Mr. Jones is filling his trading-stamp book. It holds 1200 stamps when filled. After putting in his loose stamps, Mr. Jones has 700 stamps in his book. How many more stamps does he need to fill his book?
>
> Subtraction is used to determine the size of a group within a set. "A school with 537 pupils has 249 boys. How many girls attend the school?"

Naturally you will not use such large numbers for children's first experiences with subtraction. However, children should have frequent experiences with problem situations that deal with all 4 types.

DEVELOPING READINESS FOR ADDITION AND SUBTRACTION

Activities that develop children's intuitive understanding of addition and subtraction can be an integral part of their work as they learn to count to 10. As children learn to count with concrete materials, ask questions to stimulate their thinking about addition and subtraction situations:

> "You have 6 clothespins on that card. If I give you 2 more, how many do you think you will have? Put these 2 on, and check to see if there are 8."
>
> "There are 5 acorns in your basket. If you take 3 of them from the basket, how many do you think will be left? Take out 3, and see if 2 are left."
>
> "You have 6 golf tees in the holes in that board and 4 in the holes in the other board. Which board has fewer tees in it? How can you tell? How many fewer tees does it have?"

Children can use a walk-on number line for readiness activities. When the sequence of numbers 0 through 9 has been learned, children can play train on the walk-on line. Each numeral card is a "station."

> "Begin at station 0 and walk to station 3. Now walk 4 more steps. What station are you on now?"
>
> "Begin at station 8 and walk 4 steps toward station 0. What station are you on now?"

INTRODUCING ADDITION AND SUBTRACTION

Initial Addition and Subtraction Situations

All children are not ready to be formally introduced to addition and subtraction at the same time. Those who are ready should not be delayed because of less mature classmates. When you have a group of children who are ready for this new work, introduce addition by using a concrete situation.

ACTIVITY 9-1

Introducing Addition

Begin with familiar objects such as a set of 2 dolls and a set of 3 dolls.

- Have children identify the number of dolls in each set.
- Put the 2 sets together and have the children identify the total number.
- Repeat with other objects such as toy cars, books, blocks, and paper plates.

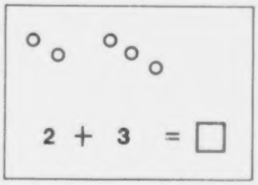
Provide many of these early experiences, which are designed to teach the meaning of addition in terms of joining sets of objects. They also offer opportunities for oral descriptions of addition situations and development of terminology: "A set of 2 dolls joined with a set of 3 dolls make a set of 5 dolls." "Two dolls and 3 dolls make 5 dolls." "Two plus 3 is 5." "Two and 3 equal 5."

After children have had a variety of manipulative experiences and can orally describe addition situations, they are ready to be introduced to the addition sentence. Materials they can move around on a magnetic or flannel board and at their desks are appropriate at this stage of the learning process.

ACTIVITY 9-2

The Addition Sentence

Put a set of 2 disks and a set of 3 disks on a flannel or magnetic board.



Introduce the addition sentence 2 + 3 = 5 by using felt or magnetic symbols. Discuss the meaning of each numeral and the + and = signs in the sentence by having children relate each one to actions with markers on the flannel or magnetic board.

Repeat with other combinations having sums less than 9.

As children use realistic objects and markers, they will build up mental images of addition situations and related numbers sentences. A child with a good store of mental images will have little difficulty in writing mathematical sentences when confronted by similar but not identical situations at a later time.

Children need not be working with all of the possible combinations of number pairs having sums through 9 before they are introduced to subtraction. Use concrete materials and begin with a take-away situation.

ACTIVITY 9-3

Introducing Subtraction

Begin with 6 books. "Here are 6 books, I am going to give 2 of them to Billy. How many books will I have left?"

Have the children count the original set to confirm that there are 6 books. Give 2 books to Billy. Many children will know that you have 4 books left. Even so, either a single child or the group in unison should count them.

Repeat with other collections of objects and different combinations.

During introductory lessons, children need not use the sentence for subtraction. Neither is it necessary to use the terms *sum, addend,* and *missing addend.* It is better first to establish an understanding of the meaning of the take-away situation. Once children recognize the meaning of take-away subtraction and can distinguish it from addition, the subtraction sentence can be introduced.

ACTIVITY 9-4

The Subtraction Sentence

Put a set of 7 markers on a flannel or magnetic board. Put a felt or magnetic "7" beneath the 7 disks.

"If I take 4 of these disks off the board, how many will be left?"

Take 4 disks off the board. After children give the answer, put the complete sentence "7 − 4 = 3" beneath the remaining disks. Relate the parts of the sentence to the concrete model. Children should see that the "7" represents the original set, the "4" represents the disks that were removed, and the "3" represents the remaining disks. Tell them that the "−" sign is read as "minus."

Repeat with other sets containing 9 or fewer objects.

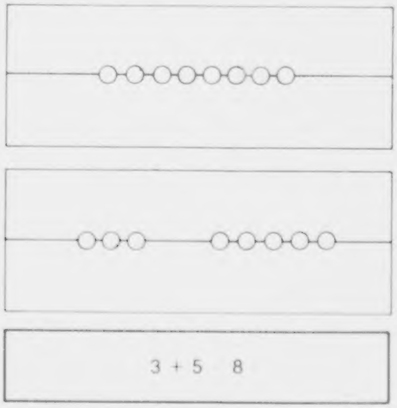
Using Manipulative Materials to Reinforce the Meaning of Addition and Subtraction

You should help children reinforce their understanding of addition and subtraction by having them engage in activities with a variety of concrete materials. Teachers who provide opportunities for children to do this report that many children need fewer drill-and-practice exercises at a later time, because of their greater understanding and because they learn many of the facts as they manipulate the objects. Activities like these are examples:

ACTIVITY 9-5

Adding with Beads

Give a child a nine-by-twelve-inch card with beads strung across the face. The child separates the beads into 2 groups and writes the addition sentence that describes the arrangement of beads.



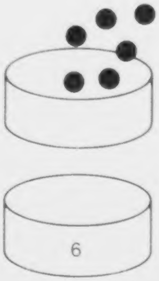
Separate the beads other ways, and use cards with other sets of beads.[1]

ACTIVITY 9-6

Adding Objects to a Container

Drop 6 objects into a margarine tub.

Write a large "6" on the tub.



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[1] Adapted from Mary Baratta-Lorton, *Workjobs* (Menlo Park, Cal.: Addison–Wesley Publishing Company, 1972), pp. 200–201.

Drop another object into the container. "How many in the tub now?" "What addition fact should we use to tell how many objects in all?" Write the sentence: "6 + 1 = 7" on the chalkboard.

Repeat with unmarked tubs and combinations with sums less than 10.[2]

ACTIVITY 9-7 Subtracting with Blocks

Give a child an aluminum pie tin and several wooden blocks.

The child chooses a number, say 6, and puts that many blocks into the tin. The child removes a few of the blocks, puts the tin upside down over them, and puts the remaining blocks on top of the tin.

The child writes 2 number sentences for the tin. If 2 blocks are covered, the sentences are 6 − 4 = 2 and 6 − 2 = 4.

Repeat with different combinations of 6 blocks and with more or fewer than 6 blocks.[3]

ACTIVITY 9-8 Dropping Beads

Enclose 7 small wooden beads in your hand. Open your hand so the children can count the beads. Close your hand.

Drop a bead from your hand onto the table. "How many beads in my hand now?" Drop another bead. "How many beads in my hand now?" "How many beads did I drop?" Confirm both numbers by having children count the beads in your hand and on the table.

Write the subtraction sentence: "7 − 2 = 5" on the chalkboard and discuss its meaning.

Repeat with other combinations of 7 and other quantities of beads.[4]

ACTIVITY 9-9 Cuisenaire Trains

Give a child Cuisenaire rods.

The child selects a rod, say a 6-rod, and makes all of the trains for 6.

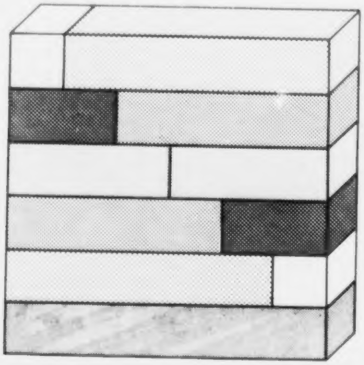
---

[2] Adapted from Larry Leutzinger, *Strategies for Learning the Basic Facts* (Cedar Falls, Iowa: Iowa Council of Teachers of Mathematics, 1981), p. 6.

[3] Adapted from Mary Baratta-Lorton, pp. 198–199.

[4] Adapted from Larry Leutzinger, p. 14.

The child writes a sentence for each train: 1 + 5 = 6, 2 + 4 = 6, 3 + 3 = 6, 4 + 2 = 6, and 5 + 1 = 6. (0 + 6 = 6 and 6 + 0 = 6 may also be written.)



- Repeat with other rods.

Activity 9-10 deals with addition and the comparison situation for subtraction.

ACTIVITY 9-10

Addition and Subtraction

- Have each child take a couple of handfuls of Unifix cubes of 4 different colors. Tell them to sort and join all cubes of the same color.
- Ask questions:
  - □ "Which color do you have most of?" Have each child hold up his or her longest link. "Who has the most?"
  - □ "Which color do you have the fewest of?" Have each child hold up his or her shortest link. "Who has the fewest?"
  - □ "How many more do you have in your longest link than in your shortest link?"
  - □ "Can you find two smaller links that together make 1 as long as the longest?"[5]

---

[5] Adapted from A. Dean Hendrickson, *Mathematics the Piaget Way* (St. Paul: Minnesota Council on Quality Education, 1980), p. 9.

Examples of real-world situations for introducing the how-many-more-are-needed and set-within-a-set situations are:

> "Sally is saving the inner seals from the lids of her favorite soft drink for a free kite. She must have 9 seals for the kite she wants. She has 6 seals now. How many more seals does she need?"
>
> "Billy watched 8 airplanes fly overhead. Before they disappeared from sight, he saw that 4 of the planes were red. How many of the planes were not red?"

Introducing the Vertical Notation Form

The mathematical sentence, rather than the vertical notation form, is most often used when children are introduced to addition. There are at least 2 reasons for this:

- It can be considered a shorthand form of the longer word sentence; hence its meaning is more easily recognized.
- It gives children a means of interpreting real-word situations that involve numbers.

However, children must also learn to use the vertical notation form because it is used in algorithms for addition, subtraction, and multiplication. The commutative property of addition (see pages 308–309) permits addends to be placed in any order without affecting the sum. However, it is less confusing to children if you are consistent in the way you write a sentence and vertical notation for a given combination. For example, for the sentence $3 + 4 = \square$, write

$$\begin{array}{r} 3 \\ \underline{+4} \end{array}$$

And for the sentence $4 + 3 = \square$, write

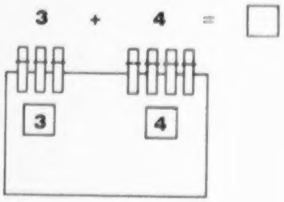
$$\begin{array}{r} 4 \\ \underline{+3} \end{array}$$

One way to introduce vertical notation is with clothespins on a card.

ACTIVITY 9-11

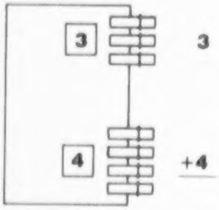
Addition Sentence Versus Vertical Form

Arrange 2 groups of clothespins along the edge of a large card.



Hold the card so the pins are along the top. Put the card against the chalkboard and write the addition sentence.

Turn the card so the pins are along the right edge; hold it against the board, and write the vertical form.



Discuss the ways the 2 forms are alike and the way they are different.

Repeat with other sets of pins.

Once children are familiar with the vertical form for addition, use a similar activity to introduce the vertical form for subtraction.

Addition and Subtraction Facts with Sums Greater Than Nine

An understanding of tens and ones in base-ten numeration is a foundation for understanding addition facts having sums greater than 9.

First, help children learn the basic facts that involve pairs of addends with a sum of 10 by using objects, markers, and charts showing patterns that make them easy to learn. Arrange markers on a flannel or magnetic board to show patterns of combinations that make 10.

After work with markers, use Activity 9-12 to prepare a chart for future reference as facts with sums from 11 through 18 are learned.

ACTIVITY 9-12

Addition Chart

Begin with a chart containing pairs of addends at the left, 10 empty circles in each line, and the numeral 10 on each line at the right.

Discuss the first combination: 10 + 0. Then have a child color 10 of the circles.

Consider the combination 9 + 1. Have a child color 9 of the circles. Discuss the fact that 9 colored and 1 plain circle make 10 circles.

Do the same for each combination until the chart is completed.

10+0	● ● ● ● ● ● ● ● ● ●	10
9+1	● ● ● ● ● ● ● ● ● ○	10
8+2	● ● ● ● ● ● ● ● ○ ○	10
7+3	● ● ● ● ● ● ● ○ ○ ○	10
6+4	● ● ● ● ● ● ○ ○ ○ ○	10
5+5	● ● ● ● ● ○ ○ ○ ○ ○	10
4+6	● ● ● ● ○ ○ ○ ○ ○ ○	10
3+7	● ● ● ○ ○ ○ ○ ○ ○ ○	10
2+8	● ● ○ ○ ○ ○ ○ ○ ○ ○	10
1+9	● ○ ○ ○ ○ ○ ○ ○ ○ ○	10
0+10	○ ○ ○ ○ ○ ○ ○ ○ ○ ○	10

The addition combinations with sums greater than 10 need to be introduced with markers and not with pictures alone. Figure 9-2 illustrates the sentence 8 + 4 = 12 with disks. Children can rearrange the disks to show that other combinations also equal 12: 9 + 3 = 12, 7 + 5 = 12, 6 + 6 = 12, 5 + 7 = 12, 4 + 8 = 12, and 3 + 9 = 12. In each case, there is a set of 10 plus 2 more disks. (The combinations 10 + 2, 11 + 1, 1 + 11, and 2 + 10 also equal 12, but are not basic facts.)

Subtraction with minuends 11 through 18 can be directly related to the addition facts with sums greater than 10. After markers have been used to show sentences such as 8 + 4 = 12 and 6 + 9 = 15, use them to show 12 − 4 = 8 and 15 − 9 = 6. Children need manipulative experiences with all combinations with sums through 18 before beginning paper-and-pencil practice activities.

FIGURE 9-2

Sets and sentences illustrating the steps in addition for 8 + 4 = ☐

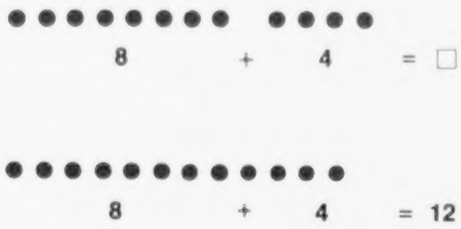
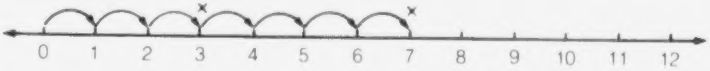


FIGURE 9-3

Number line showing how to illustrate the sentence 3 + 4 = 7



Using the Number Line to Learn Facts

A chalkboard number line provides a useful way to reinforce children's understanding of addition and subtraction and to build mental images of the operations. Some teachers use the idea of a cricket or frog jumping along the line to introduce a chalkboard number line. To illustrate the sentence 3 + 4 = ☐, children can see the cricket or frog start at 0 and take 3 jumps, and then jump four more units to 7, as shown in Figure 9-3. The "X" indicates stops at 4 and 7.

The number line is also useful for showing the relationship between addition and subtraction. First, use an addition sentence. The sentence 6 + 3 = 9 is illustrated in Figure 9-4(a). Children should think of ways the number line can show subtraction. Those who have suggestions should have an opportunity to present their ideas to the class. The procedure illustrated on the number line in Figure 9-4(b) follows logically from the use of the "jumps" in addition. Children should note that the jump showing the number to be subtracted begins at the point that indicates the number from which it is to be subtracted and moves to the left. The fact that subtraction jumps move to the left and addition jumps move to the right should be stressed.

The number line can help children understand the meaning of basic addition facts with larger sums. A sentence, such as 7 + 8 = ☐, should be displayed. Call a child to the number line to show the first addend using cricket jumps. Then the second addend should be shown with a second set of 8 jumps (Figure 9-5). The addition sentences should be written on the chalkboard as the steps are shown on the number line.

7 + 8 = ☐

FIGURE 9-4

Number lines illustrating the inverse relationship between addition and subtraction. (a) The addition sentence 6 + 3 = 9. (b) The subtraction sentence 9 − 3 = 6.

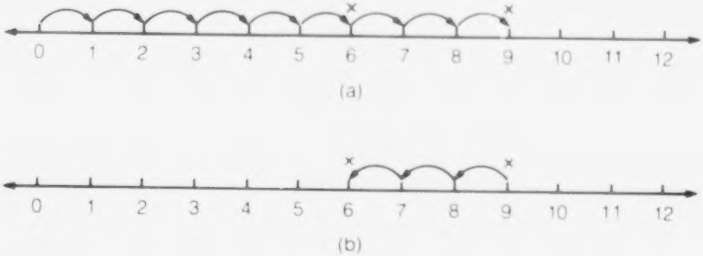
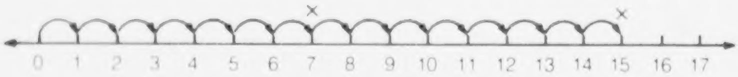


FIGURE 9-5

Number line showing the addition 7 + 8 = ☐



REINFORCING THE LEARNING OF BASIC FACTS

All children who are capable of doing so should learn the basic facts for rapid recall. Rather than giving children random practice-and-drill activities, you should help them by presenting strategies for understanding and remembering the facts. Several effective strategies for second, third, and fourth graders are presented here.

Using Families to Learn the Facts

Addition and subtraction facts can be organized by *families*. Addition and subtraction facts are grouped by families in these 2 ways:

> All sentences for a given sum are grouped together: 3 + 0 = 3, 2 + 1 = 3, 1 + 2 = 3, 0 + 3 = 3, and 3 − 0 = 3, 3 − 1 = 2, 3 − 2 = 1, 3 − 3 = 0.
>
> Or, the 4 combinations involving a given pair of addends and their sum are grouped together: 6 + 2 = 8, 2 + 6 = 8, 8 − 2 = 6, and 8 − 6 = 2.

Using Doubles and Near Doubles

Doubles are combinations with like addends; near doubles are combinations with numbers that differ by 1 or 2. Most students have little difficulty learning the doubles. Once these have been learned, the near doubles can be learned.

ACTIVITY 9-13

Near Doubles

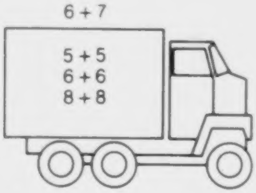
Have each child make a set of double cards.

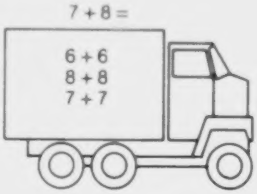
4 + 4

5 + 5

6 + 6

- Hold up a near-double card, such as one with the combination 5 + 6. Ask, "Which double card will help us with this addition?" Some children will hold up 5 + 5, while others may hold up 6 + 6. The first card shows a "double plus 1" combination, while the second shows a "double minus 1" combination. Either strategy is a useful one, and children should be encouraged to consider both of them.
- Repeat with other combinations of numbers.
- Reinforce with a work sheet containing examples like this: "Ring the doubles you could use to solve the combination on each truck."[6]





3 Adding One Less Than Ten

Children's knowledge of adding a single number to 10 can be used to help learn addition with 9. Orally present combinations with 10, such as 10 + 3, 10 + 8, and 10 + 9. Follow with this activity where children add one less than 10:

ACTIVITY 9-14

Adding One Less Than Ten

Present oral problems. The first should contain a 10 as one addend; the second has the 10 replaced by 9: "10 + 5 = ? (15) 9 + 5 is one less than 10 + 5, so 9 + 5 = ?" (14) "10 + 7 = ? (17) 9 + 7 is 1 less than 10 + 7, so 9 + 7 = ?" (16). After the pattern is established, practice the 9 facts without reference to 10.[7]

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[6] Adapted from Larry Leutzinger, pp. 24–25.

[7] Adapted from Larry Leutzinger, p. 29.

Using the Addition Table to Learn Facts

Sometime after all the basic facts have been introduced, children should see the facts organized in a table (Figure 9-6). The table should not be presented already filled in; neither should the children simply fill it in as a seatwork activity. Rather, children should complete it as part of a group lesson during which you discuss with them features that will help them remember the facts. Among the ideas to be stressed are these:

FIGURE 9-6

An addition table helps children learn the addition facts.

+	0	1	2	3	4	5	6	7	8	9
0										
1										
2										
3										
4										
5										
6										
7										
8										
9										

(a)

+	0	1	2	3	4	5	6	7	8	9
0	0	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9	10
2	2	3	4	5		7				
3	3	4	5	6	7	8				
4	4	5		7	8	9				
5	5	6	7	8	9	10	11	12	13	14
6	6	7				11	12	13		
7	7	8				12	13	14	15	
8	8	9				13		15	16	17
9	9	10				14			17	18

(b)

1. The first row at the top and the first column at the left contain facts involving 0—the identity for addition. (Ways of teaching the addition identity are discussed later in this chapter.) These facts are easily mastered.
2. The second row and second column contain sums for facts involving 1 as an addend. The sums in each row and column result from adding 1 to the other addend.
3. Sums for facts involving 5 as an addend are in the sixth row and sixth column.
4. The diagonal from the upper-left to lower-right corners contains the sums of the doubles, that is, pairs of like addends.
5. The diagonals immediately above and below the one containing doubles contain the near doubles. Sentences like $5 + 6 = 11$ and $7 + 8 = 15$ are near doubles. Once children have mastered the doubles, they can use them to learn the near doubles.
6. By now, most of the table has been filled in and there remain only a few facts to consider (see Figure 9-6). These are the combinations that give children the most difficulty. Most children will be impressed with how few difficult combinations there are. These should be identified and included in future practice activities.
7. Because subtraction is the inverse of addition, the table also contains the subtraction facts. Instead of using the numbers in the top row and left column as addends to determine sums, use a sum and addend to find a missing addend. To find the answer to $9 - 6 = \square$, locate 9 opposite 6 in the left column and go to the top of the column containing 9; the answer—3—is located at the top of the column. For $9 - 3 = \square$, locate 9 opposite 3 in the left column and go to the top of the column to locate the answer—6.
8. Point out the roles of 0 and 1 in subtraction; discuss the doubles and near doubles in a manner similar to the discussion of them in addition; this will help children master them for subtraction.

PRACTICING THE BASIC FACTS

Children's first experiences with addition and subtraction should be with activities that develop understanding of the operations and present the basic facts. Remembering the facts is not a significant part of the initial learning, but when you are sure that children understand the meanings, provide opportunities for them to learn each fact for rapid recall.

Five generalizations regarding practice in mathematics are given here:

1. Practice and understanding should go hand in hand.
2. The reasons for practice must be clear to children.
3. The kind and amount of practice are not likely to be the same for all children at the same time.
4. The practice sessions should be brief and occur often.
5. A variety of materials and procedures should be used.

In addition to these generalizations, which govern the way practice sessions are organized, there are principles that are helpful for guiding specific strategies during practice sessions.* During practice sessions, stress rapid recall. If children hesitate when shown a flash card or given a combination orally, do not give an explanation or allow time for them to count on their fingers. Instead, show or name the answer. Then show the flash card, or repeat the combination orally, and have the child respond immediately. Provide practice with only a few combinations at a time; give frequent review of previously learned combinations. Provide constant encouragement and recognition of progress through verbal praise and individual record charts.

Practice materials are available from many sources. Most commonly used are the practice pages from workbooks. These materials have a place in the program but should not be used to the exclusion of other materials. Preprogrammed devices such as the one illustrated in Figure 4-3 provide many practice activities.

You can choose from several available computer programs to emphasize learning strategies and practice of basic facts in addition and subtraction. In *Stickybear Math*[9], children help the Stickybear character move through an adventure by answering problems. The first of 20 levels of difficulty involves 1-digit problems. Both horizontal and vertical notation are used. For *Number Stumper*[10], children practice addition and subtraction to 12 by viewing their available number bank and comparing it to the sum of dice on the screen. *Addition Magician*[11] provides practice in addition from 6 to 18. Strategy is also needed, however, because children must find 3 adjacent numbers on a grid that give the desired sum. *Math Blaster*[12] is pure drill and practice in an arcade-game format with an emphasis on speed.

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* These principles are adapted from Edward J. Davis, "Teaching the Basic Facts," *Developing Computational Skills*, 1978 Yearbook of the National Council of Teachers of Mathematics, 1978, pp. 52–58.

[9] Weekly Reader Family Software, 245 Long Hill Rd., Middletown, CT 06457.

[10] The Learning Company, 6493 Kaiser Drive, Fremont, CA 94555.

[11] The Learning Company.

[12] Davidson & Associates, 3135 Kashiwa St., Torrance, CA 90595.

Graphics, sound, and avoidance of negative feedback have contributed to *Math Blaster*'s immense popularity as an educational program. Children may choose problems from addition, subtraction, multiplication, or division, and a problem editor allows individualizing practice problems for a student or group.

Appendix B contains names and addresses of companies to which you can write for catalogs listing new programs to review and evaluate.

Calculators can also be used for practicing basic facts. Activity 9-15 not only provides practice but also points out an advantage of knowing the facts well. In Activity 9-16, students begin working with simple combinations, but this activity can also be extended to 3 or more addends and other combinations.

ACTIVITY 9-15

Brain versus Button

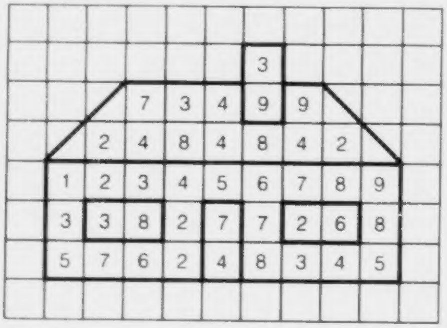
- Use this activity after children are proficient with basic facts of addition and subtraction.
- Divide the class into pairs. One person in each pair is the Brain who will work problems mentally and write the answer on paper. The other will be the Button who must key in all the problems on a calculator, then write the answer.
- Ask the children who they think will win.
- Start the race and time how long it takes the Brain and how long it takes the Button. Then let the children trade roles. Record the times on the chalkboard and compare.
- You may want to consider accuracy in the contest by setting a time penalty for wrong answers.
- Variation: You can conduct the same activity for older children. Start with a race on basic facts, then progress to addition and subtraction problems with 2- and 3-digit numbers or addition with 4 or 5 addends. Ask the children who they think will win in each race. This activity alerts children to strengths and weaknesses of the calculator. Intermediate students can also use this activity to generate data for charts or calculating averages.

ACTIVITY 9-16

House Numbers

- Draw a simple picture, such as a house, on centimeter-square paper. Make the door, windows, and chimney 2 squares each. Write a digit 1–9 in each square inside the house.

- Use the calculator to total the numbers inside the door, windows, and chimney.



						3				
			7	3	4	9	9			
		2	4	8	4	8	4	2		
	1	2	3	4	5	6	7	8	9	
	3	3	8	2	7	7	2	6	8	
	5	7	6	2	4	8	3	4	5	

- Variation:
(a) Prepare a task card that contains the house and additional problems like these:
The sum of numbers in the door and the chimney is ____________ .
The sum of the numbers in the chimney and the left window is ____________ .
The difference between the sum of the windows and the sum of the chimney is ____________ .
(b) Find the sum of the numbers in the roof; in the wall outside the door and windows.
(c) Find the sum of all the house numbers.[13]

Games provide another source of activities. Addition bingo is an example of a large group game. Each playing card is different. A number (sum) may appear on a game board as many times as it appears on the addition table (see Figure 9-6). Bingo cards for each operation are available commercially or may be made by a teacher.

GAME 9-1

Addition Bingo

Materials: Game board for each child, markers, addition facts sheet or flash cards.

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[13] Adapted from Gary Bitter and Jerald Mikesell, *Activities Handbook for Teaching with the Hand-Held Calculator* (Boston: Allyn and Bacon, 1980), 68–69.

- Give each child 1 bingo card and about 20 markers.

0	6	18	9	15
4	3	11	8	7
5	7	Free	17	16
6	15	13	2	4
10	1	12	14	6

- One at a time, read a combination from the fact sheet or show one of the flash cards. Children cover the numeral that names the sum for the combination. Only 1 numeral is covered for each combination, even when the sum appears more than once on a board.
- A winner is declared when a child has 5 markers in a row, column, or diagonally. A game of "blackout" requires that all 25 squares be covered.
- Variation: Make a bingo board with the problems (6 + 2, 4 + 3, and so on) on it. Hold up answer cards or call out sums; children cover a problem square that matches each sum.

Activity 9-17 is a good one for developing mental computation skills. Children must listen carefully to determine what rule applies to the numbers given; and to the replies that you or a classmate give in response to the original numbers. Then they must apply that rule to new numbers as they are given.

ACTIVITY 9-17

What's My Rule?

- Tell the children that you are going to pretend that you are a "rule" machine. This special machine is able to do things to numbers. When a number is given, the machine will apply its rule to change the number and give back another number. If children are learning addition, make the machine an "add" machine. (It could also be a "subtract," "multiply," or "divide" machine.) Children are to listen to a number as it is given and to your response to determine what the machine (you) are doing to each number to change it.
- Keep the rule simple at first. An "add-2" rule is good to begin with.
- Tell the children to give you a number from 0 to 9. When a number is given, for example 4, you think "4 + 2 = 6" and say "6." If 8 is given, you say "10." You never state the rule, only the answer.

- When children think they know your rule, they are not to say what it is but should raise their hands. When you see raised hands, invite those children to join with you to give the next response. If their responses are correct, you will know that they know the rule. If an answer is incorrect, tell the child who gave it to listen again and rethink the rule.
- Variation: Write a short Logo program that has a mystery rule in it.

```
TO MYSTERY.RULE :NUMBER
OUTPUT :NUMBER + 5
END
```

When a child types PRINT MYSTERY.RULE 6, the computer prints out 11. After typing in several different numbers to the mystery rule and observing the output, children should be able to guess what the rule is. They can soon begin to write their own procedures with rules and challenge friends to guess their rule.

```
TO JOHN.RULE :NUMBER
OUTPUT :NUMBER + :NUMBER - 3
END
TO STACY.RULE :NUMBER
OUTPUT :NUMBER + :NUMBER/2
END
```

TEACHING SOME PROPERTIES OF ADDITION AND SUBTRACTION

The commutative and associative properties and identity element for addition need not be stressed in highly formal ways; however, their meanings and applications should be developed informally.

The Commutative Property

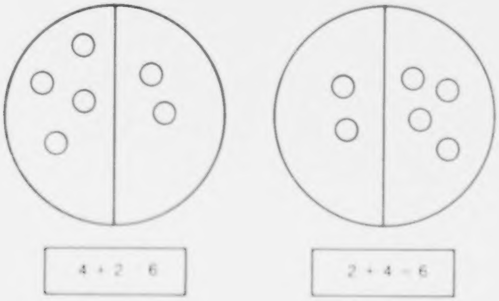
The following procedure is one of the many effective ways to informally introduce the commutative property.

ACTIVITY 9-18

Commutative Property

- Give each child a paper plate that is divided into halves with a line, and a handful of markers.

- Have each child put a few markers in each half of the plate. Each child writes the addition sentence for the markers on the plate.
- Have each child turn the plate halfway around, so that the two sets of markers are reversed. Each child writes a new sentence for the markers.



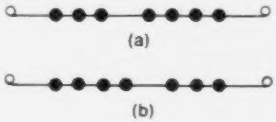
- Discuss each of the sentences.

Other examples of how to help children learn the commutative property of addition are:

- Put 2 sets of clothespins on the edge of a large card. Hold the card against the chalkboard and write the sentence, including the sum, above the card. Turn the card so the order of the sets is reversed and write the new sentence.
- Use beads on pieces of stiff wire like those shown in Figure 9-7. Show the sets (a), and then reverse their order (b).

FIGURE 9-7

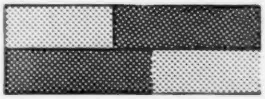
Beads on a wire to show sentences (a) 3 + 4 = ☐ and (b) 4 + 3 = ☐



- Use Cuisenaire-rod trains. Put a train of a 3-rod and a 4-rod next to one made of a 4-rod and a 3-rod (Figure 9-8).

FIGURE 9-8

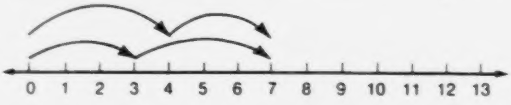
A Cuisenaire-rod train illustrates the commutative property.



- Use a number line such as the one shown in Figure 9-9.

FIGURE 9-9

Number line used to show the commutative nature of addition



The Associative Property

The associative property of addition is used when 3 or more addends are added. Because addition can be done with only 2 addends at a time (it is a binary operation), it is necessary to select 2 of the 3 addends to add first; then the third number is added to the sum of the first 2.

Addition using 3 addends is usually introduced before all the 100 basic addition facts have been introduced. No specific number of basic facts is a prerequisite, but children must previously have worked with all the basic facts they will use when they first work with 3 addends. The sum of the first pair of addends should be less than 10. As with other new topics, this one should be introduced by activities with sets of manipulative materials. During their first experiences with 3 addends, children should add them in order, reading from left to right.

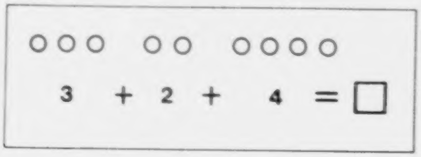
- Use clothespins on a card. Put 3 sets of pins on 1 edge of the card. Place the card against the chalkboard, and write the addition sentence above the pins. Combine the pins in the two sets to the children's left; then unite the third set with the combined set.
- Use sets of beads on a wire, following the procedure for clothespins on a card.
- Use Cuisenaire rod trains. Make a 3-car train. Have children consider first the 2 rods on the left and then the third rod.

After children have learned to add in this manner, the associative property of addition can be introduced through a situation such as this:

ACTIVITY 9-19

Three Addends

Arrange 3 small sets horizontally across a magnetic or flannel board, and put the mathematical sentence describing the situation beneath the markers.



Have a child join the first 2 sets of markers, and have the children think of the resulting addition sentence.

$$3 + 2 = 5$$

Put this sentence beneath the original sentence. Have the children note that there is a set of 4 to be joined with the new set of 5. Put the new sentence $5 + 4 = \square$ beneath the others.

When all of the steps are completed, these sentences will be on the board:

$$\begin{aligned} 3 + 2 + 4 &= \square \\ 3 + 2 &= 5 \\ 5 + 4 &= \square \\ 5 + 4 &= 9 \end{aligned}$$

Review the meaning of each of the sentences.

Arrange the sets as they originally appeared on the board.

Have a child join the second and third sets, and have the children discuss the resulting addition sentence, $2 + 4 = 6$. Put this sentence beneath the original sentence. Have the children note that there is now a set of 3 to be joined to the set of 6. Put the new sentence $3 + 6 = \square$ beneath all the others.

When all of the steps are completed, these sentences will be on the board:

$$\begin{aligned} 3 + 2 + 4 &= \square \\ 2 + 4 &= 6 \\ 3 + 6 &= \square \\ 3 + 6 &= 9 \end{aligned}$$

Review the meaning of each sentence. Then compare them with the first group of sentences.

Repeat with other sets.

Children in the third grade and beyond can refine their thinking about the 2 properties through work with examples like these:

$$246 + 398 + 2 = \square$$

Consider the sum for this example without using paper and pencil. Do the children realize that they can add the 398 and 2 first to get the sum 400, and then add the 246 to 400? Other examples can combine the 2 properties:

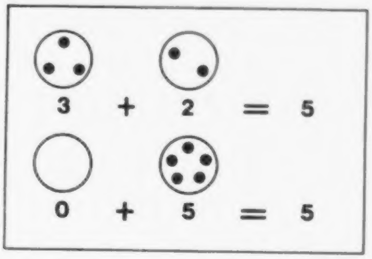
$$\begin{aligned} 696 + 241 + 4 &= \square \\ 10 + 367 + 590 &= \square \\ 16 + 83 + 7 + 4 &= \square \end{aligned}$$

The Identity Element of Addition

Children who understand the meaning of 0 will have little difficulty learning its role in addition. As they deal with markers to learn about addition,

FIGURE 9-10

Sets illustrating addition sentences 3 + 2 = 5 and 0 + 5 = 5



they will realize that when 0 is an addend, there will be no markers representing that number. To make this more obvious, some primary-grade teachers prefer clearly identifying each group by putting a ring around the markers on a display board. The two addition sentences illustrated in Figure 9-10 are represented by ringed markers. The ring above the 0 in the second sentence indicates the absence of markers when 0 is an addend.

Eventually children will see that when 0 is one of a pair of addends, the sum is always the number of the other addend. Zero is the identity element for addition.

During work with subtraction, children should be given opportunities for the role of 0 to become clear to them. They should see that when 0 is subtracted from another number, the answer is always the other number. And when a number is subtracted from itself, the answer is always 0.

TEACHING HIGHER-DECADE ADDITION

Higher-decade addition is adding a number smaller than 10 to a number between 9 and 100. Examples are 24 + 6 = 30 and 45 + 9 = 54. The importance of being able to add such numbers in a single step, with no regrouping, is evident when you consider these situations:

> You are adding 8 + 9 + 7 + 8. After you add 8 and 9, you add 7 to 17 and then you add 8 to 24.
>
> You are multiplying 96 × 48.

$$\begin{array}{r} 48 \\ \underline{\times 96} \end{array}$$

After you multiply 6 times 8, you record "8" and regroup 4. The product of 6 times 4, 24, is added to the 4 that you regrouped. As the multiplication of 90 × 48 is completed, you add 7 to 36.

In both cases a person who can complete addition of the 2 numbers in 1 step has an advantage over one who must do the addition in stages. Children need special instruction in adding these numbers.

Skip-counting, which is discussed in Chapter 7, contributes to children's background for this addition; but it will not lead to a high level of proficiency, because it deals with patterns rather than random situations, as occurs in the previous examples of addition and multiplication. Third- and fourth-grade children should have introductory and practice activities like the following so they will understand higher-decade addition:

ACTIVITY 9-20 Higher-decade Addition

Use a hundreds chart (see Activity 7-6).

For a sentence like 45 + 6 = □, call the children's attention to 45 on the chart; then have them count on 6. "Where did the count stop?" "If you begin at 55, where will the count stop?" "What is 65 + 6? 75 + 6? 85 + 6?"

Repeat with other numbers.

Variation: Use a number line rather than a hundreds chart.

After children understand higher-decade addition, they need practice so they can do it as a 1-step process. Here are samples of both written and verbal exercises:

List addition sentences on a work sheet according to a pattern:

5 + 3 = □, 15 + 3 = □, . . . 95 + 3 = □
7 + 8 = □, 17 + 8 = □, . . . 87 + 8 = □

List addition sentences on a work sheet in random order:

14 + 9 = □, 35 + 7 = □, 64 + 8 = □

Dictate sentences to which children give oral responses: "Add 6 to 45." "Add 4 to the product of 8 and 7."

TEACHING ADDITION WITH LARGER WHOLE NUMBERS

Children who have a good understanding of whole numbers and place value are ready to start adding tens and hundreds numbers. This work typically begins in the late first grade or second grade. A first lesson should begin with a common problem situation and proportional place-value models such as a place-value pocket chart or beansticks with which children are familiar.

ACTIVITY 9-21

Adding Tens

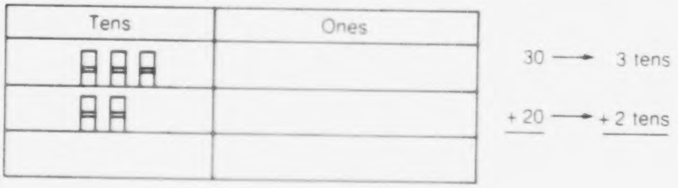
- Tell a story. "Yesterday John had 30 baseball trading cards. He got 20 more today. How many cards does he have now?"
- Allow time for children to determine the answer by individual methods.
- When children have determined their answers, write the addition sentence:

$$30 + 20 = 50$$

- Write the same problem in vertical form:

$$\begin{array}{r}30\\ \underline{+20}\end{array}$$

- Have a child represent the addends in a place-value pocket chart.



- Have another child combine the 3 tens bundles and the 2 tens bundles and put them in the chart's bottom pocket. Discuss the fact that there are a total of 5 tens, or 50.
- Put the answer in the algorithm.
- Repeat with similar situations and numbers until the process is clear.
- Variations: Children can show the 3 tens and the 2 tens with beansticks, Cuisenaire rods, Unifix cubes, or base-ten blocks.

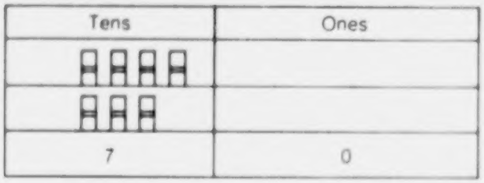


FIGURE 9-11

Child's work sheet showing that 40 + 30 = 70

In lessons that follow, children can use work sheets that contain pictures of the place-value pocket chart or another place-value model. Instead of using markers (concrete mode), they use pictures of them (pictorial mode). Figure 9-11 illustrates an example of the type of work children might complete on a work sheet. To illustrate 40 + 30 = ☐, a child draws markers to show the 4 tens in 1 addend and the 3 tens in the other. Then the child writes numerals at the bottom to indicate the total number of tens—7—and ones—0. Six or eight examples can be put on 1 work sheet.

Addition of only multiples of 10 will not present any special problems to most children. This should be soon followed by addition of numbers without 0 in the ones place. Most textbooks introduce the addition of these numbers with examples that do not require regrouping, or "carrying," as the numbers are added. Some authorities believe children should begin regrouping right away. Their argument states that when children add 2 numbers, such as 23 and 45, using the conventional algorithm form

$$\begin{array}{r} 23 \\ +45 \\ \hline \end{array}$$

they can begin their addition in either the tens or the ones place and get the correct sum either way. Thus they do not learn the value of beginning addition in the ones place. If an addition requires regrouping, as

$$\begin{array}{r} 28 \\ +47 \\ \hline \end{array}$$

does, they will learn from the beginning that it is convenient to start the addition in the ones place and then go to the tens place. You will probably decide which kind of example to use on the basis of what the textbooks used in the classroom include. This is reasonable in most cases, because the textbook will provide much of the practice materials the children will use once they have been introduced to the new work. Of course it is not necessary to introduce a topic according to the text's order of presentation. If your children have a good understanding of numbers and the simpler addition processes, you may choose to use examples that require regrouping regardless of the text's order of progression. Then addition practice with and without regrouping can proceed as you select material from the textbook as needed.

When addition with regrouping is introduced, proportional place-value materials, such as bundled tongue depressors, Unifix cubes, and beansticks are useful for making regrouping meaningful. The following situation might begin a lesson:

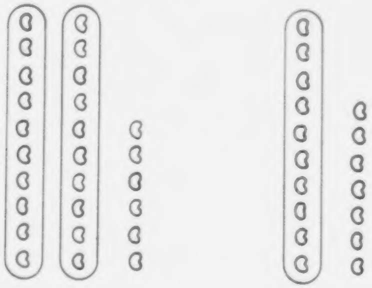
ACTIVITY 9-22

Regrouping

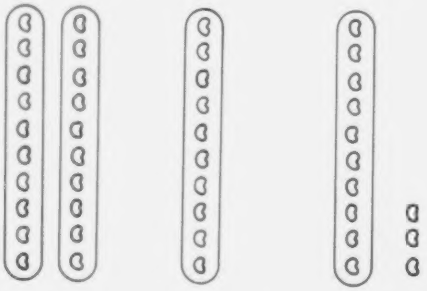
- Tell a story. "Sally has 26 marbles. Her grandmother brings her 17 more. How many marbles does she now have?"
- Have the children determine the addition sentence for this problem

$$26 + 17 = \square$$

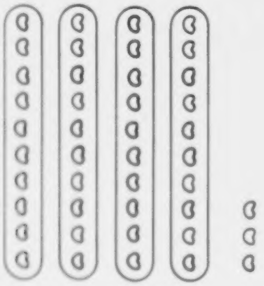
- Have each child show the 2 numbers with beansticks.



- Each child combines the loose beans and determines how many there are. "Are there more than 9 loose beans?" (Yes.) "What do you do if there are more than 9 loose beans?" (Exchange 10 of them for a tens stick.) Have each child do this.



- Each child combines the tens sticks. "How many tens sticks do you have?" (There are 4 tens sticks.) "How many tens and how many ones do you have?" (There are 4 tens and 3 ones.)



Repeat with similar stories and numbers.

Children should have enough activities with place-value materials to make the process clear before an algorithm is introduced. When you introduce an algorithm, relate each of its steps to each step on the place-value model. Using the problem in Activity 9-22 as an example, children will work with this algorithm.

$$\begin{array}{r} 26 \\ \underline{+17} \end{array}$$

They should associate the addition of 6 and 7 with the combining of the loose beans. The 3 in the ones place is associated with the 3 loose beans. The 1 that is regrouped in the algorithm should be associated with the exchange of 10 loose beans for 1 tens stick; and the addition of the 1, the 2, and the 1 should be associated with the combining of the tens sticks.

Some children are helped by seeing the algorithm's numerals expressed in expanded notation:

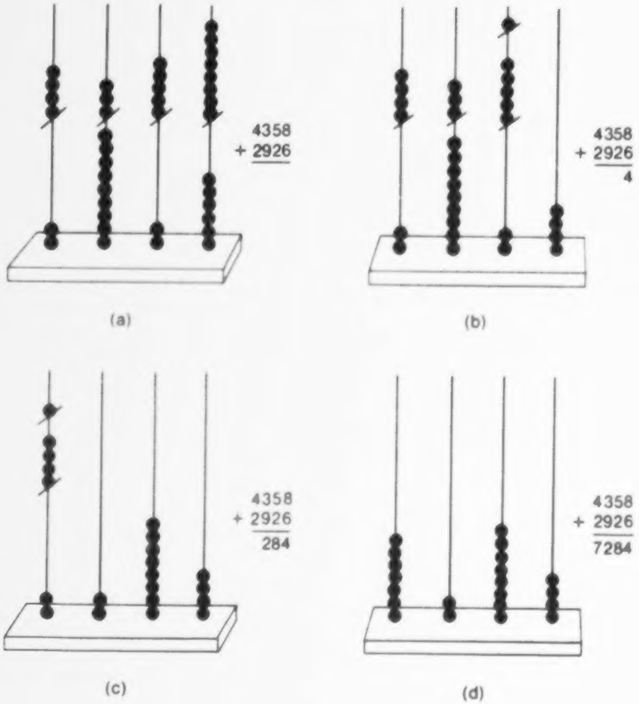
$$\begin{array}{rcl} 26 & = & \text{2 tens and 6 ones} \\ \underline{+17} & = & \underline{\text{1 ten and 7 ones}} \end{array}$$

Now when they add 6 and 7, they see that they have 13 ones. When 10 of these are regrouped as 1 ten, they can add the 10 to the 2 tens and 1 ten, giving a total of 4 tens.

As children proceed through the grades, you should review the meaning of regrouping with larger numbers. A good way to do this is with an abacus. This nonproportional device can show the larger numbers and illustrate each step in an algorithm.

FIGURE 9-12

Abacus illustrating steps in addition. (a) The two addends. (b) Addition and regrouping in the ones place. (c) Addition in the tens place and addition and regrouping in the hundreds place. (d) Addition in the thousands place.



To use the abacus to illustrate addition, put an example such as

$$\begin{array}{r} 4358 \\ +2926 \\ \hline \end{array}$$

on the chalkboard and go through the steps shown in Figure 9-12. Ask a child to set beads on the abacus to represent the addend in the lower portion of the algorithm, 2926. Then the other addend, 4358, can be represented with beads at the top of the abacus. In the figure, small clips, such as tiny plastic clothespins, separate the beads on the wires to illustrate the 2 addends. The abacus would appear as shown in (a). The meaning of the beads and their relationship to the addends in the algorithm should be discussed. Next, a child should unite the beads on the ones wire by removing the clip between the 2 sets (b), while a second child works with the algorithm on the board. The regrouping of the sum in the ones place is demonstrated as 10 beads from the ones wire are exchanged for 1 bead on the tens wire. (The 1 bead on the tens wire is separated from the others by a clip.) The beads on the tens wire are then combined, and the numbers in the tens place of the algorithm are added. The process continues as the beads on the hundreds wire are united. Again the regrouping is shown as 10 beads from the hundreds wire are exchanged for 1 bead on the thousands wire (c). The process is completed as the beads on the thousands wire are combined and the algorithm is finished (d). The addition of pairs of numbers having sums to 9999 can be shown in a 4-wire abacus. For larger numbers an abacus with more wires is needed.

ESTIMATING SUMS

As children are learning to add larger numbers, they should also learn skills that will enable them to make reasonable estimates of sums for 2 or more numbers. There are several processes, such as rounding-off and adding the rounded numbers, using the compatible numbers strategy, and front-end estimation. Procedures for teaching children to round numbers are presented in Chapter 5. It is recommended that estimation activities be introduced in 2 stages. During stage 1 children estimate sums for each of the examples in an exercise. For example, with addition such as

$$\begin{array}{r} 55 \\ 78 \\ 42 \\ 11 \\ 23 \\ +86 \\ \hline \end{array}$$

a child can use the rounding-off process to estimate the answer. During this stage, a child rounds each number to the nearer 10 and uses the numbers in the tens place to estimate a sum for 5 to 10 different examples. When children have completed their estimates, you will want to check the reasonableness of estimated sums by collecting and reviewing all of the papers or by listing all estimates on the chalkboard and reviewing them with the children. The numbers are actually added during the second stage. Unless the 2 stages are clearly separated, many children will compute the sum for each set of numbers first and use that sum to determine an estimated sum. If the sum is 295, for example, a child who adds the numbers first would round 295 to the nearer hundred and give an answer of 300. No estimation has been done and there is no benefit to the child who does this. Children need to be impressed with the idea that estimates are used to judge the reasonableness of a computed answer, or to determine an approximate answer when an exact answer is unnecessary.

Compatible numbers are numbers that "fit" together because they are easy to deal with mentally. You can help children form compatible pairs of numbers by having them examine sets of numbers such as these:

45, 72, 25 278, 935, 756 6723, 3279, 8472

In the first set, children are to find the pair that fits together to make approximately 100; in the second set, what pair fits to make about 1000? In the third set, what pair fits to make about 10,000? The first set of numbers is suitable for third or fourth graders, while children in any intermediate grade should be able to cope with the second and third sets. When children demonstrate that they are able to form compatible pairs, they can use the strategy to estimate sums. In the example used earlier (see margin), the first and third, second and fifth, and fourth and sixth pairs of numbers each fit to make about 100. Since there are 3 pairs of numbers, an estimate of the sum is 300.[14]

55
78
42
11
23
+86

Front-end estimation is based on the idea that numbers in high place-value positions are more significant in an answer than are numbers in the small positions. It can be used with all 4 operations but is most useful with addition. Third- and fourth-grade children can use front-end estimation with examples like this:

125
345
226
45
+152

The numbers in the hundreds place are added first, giving 700. Then the tens and ones are mentally added: 25 and 45 and 26 are about 100, and 45 and 52 are about 100, giving 200 more, which is added to 700, giving an estimate of 900.

To estimate a sum for these four numbers

5267
8942
372
+1217

the numbers in the thousands places are mentally added, giving 14,000. Then the numbers in the hundreds places are mentally added, giving 1600.

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[14] For more information about compatible numbers, see Barbara S. Reys, "Teaching Computational Estimation: Concepts and Strategies," in *Estimation and Mental Computation* (Reston, Va.: National Council of Teachers of Mathematics, 1986), p. 41.

The 1600 is added to 14,000, giving an estimated answer of 15,600. With larger numbers like these, the tens and ones places are disregarded.

Once children have been introduced to an estimation strategy, they should practice and apply it. It is not always necessary that exact answers be determined. Sets of problem cards, practice sheets, and word problems involving addition, either in learning centers or otherwise accessible to children, can provide frequent practice opportunities.

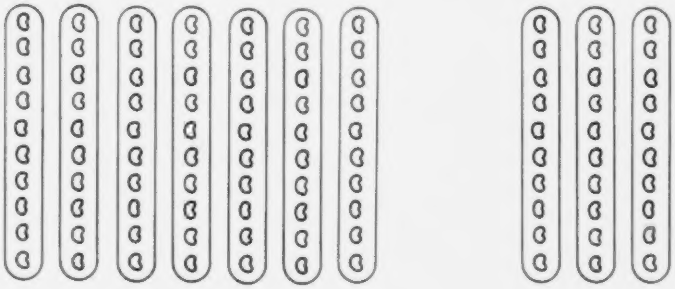
INTRODUCING SUBTRACTION WITH LARGER WHOLE NUMBERS

Introductory work with subtraction of larger whole numbers should parallel introductory work for addition. That is, first work should be with realistic situations and adequate time for children to explore the operation's meaning. "A rope 70 feet long is being cut into smaller jump ropes. If 40 feet of rope have been cut from it, how many feet remain to be cut?" After children have used their own methods to determine the answer, have different ones share their procedures. Place-value models should be brought into the discussion either by a child or by you. Beansticks are illustrated here, but any of the models can be used. In Figure 9-13 (a), the 70 feet of rope is represented by 7 tens sticks. The removal of 40 feet of rope is indicated by removing 4 of the tens sticks. The answer, 30 feet of rope, is represented by the 3 tens sticks that remain (b).

Other similar problem situations should be illustrated with models until the meaning of the subtraction is clear. The children should also represent situations by drawing markers on place-value models illustrated on work sheets. Figure 9-14 illustrates an example of how a child can show the subtraction 80 − 50 = ☐ on a picture of a place-value pocket chart. In (a), the 80 is represented by 8 bundles of 10 markers drawn on the chart. In (b),

FIGURE 9-13

Beansticks showing the meaning of 70 − 40 = 30

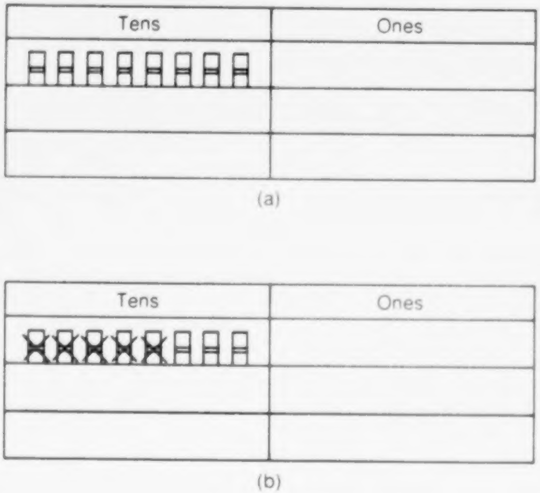


(a)

(b)

FIGURE 9-14

Child's work sheet showing that 80 − 50 = 30



(a)

(b)

5 bundles have been marked out to indicate the subtraction of 50, leaving the 3 bundles that represent the answer. The algorithm form

$$\begin{array}{r}60\\ -20\\ \hline\end{array}$$

should be introduced only after children indicate that they are ready for it. Check the children's understanding from time to time by having them illustrate further examples with place-value models and drawings of them.

TEACHING SUBTRACTION WITH REGROUPING

Subtraction with regrouping has been the subject of much debate: Which of 2 methods of computation should children be taught?[15] The decomposition method is most commonly used, but there are many advocates for the equal-additions method.

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[15] Actually there are more than 2 processes. Not commonly used in school mathematics programs and subject to less debate are those described by Hitoshi Ikeda and Masui Ando, "A New Algorithm for Subtraction?" *Arithmetic Teacher*, XXI, No. 8 (December 1974), pp. 716–719; and by Paul R. Neureiter, "The 'Ultimate' Form of the Subtraction Algorithm," *Arithmetic Teacher*, XII, No. 4 (April 1965), pp. 277–281.

The Decomposition Method

The decomposition method of subtraction is generally favored because children easily rationalize its process of regrouping as they use a place-value model. The relationship between subtraction and addition is emphasized by this method, because the steps of subtraction by decomposition are the reverse of those for addition by regrouping.

A good understanding of place value is a prerequisite of the decomposition method of subtraction. An understanding of expanded notation is also useful. For example, children should know that the number 36 can be expressed as "36," "3 tens and 6 ones," "30 + 6," and "20 + 16."

The following activity will introduce children to subtraction using decomposition:

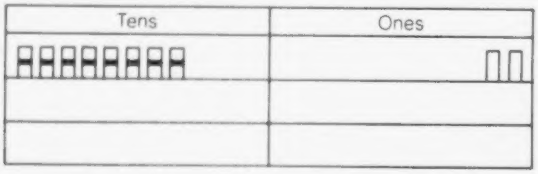
ACTIVITY 9-23

Decomposition

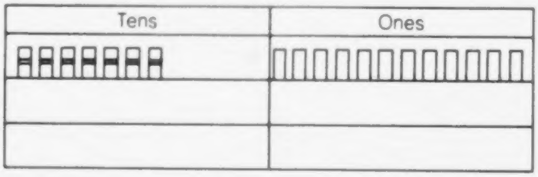
- "There were 82 boxes of apples at a fruit stand when it opened. During the day 53 boxes were sold. How many boxes were left at the end of the day?"
- Children should determine the mathematical sentence:

$$82 - 53 = \square$$

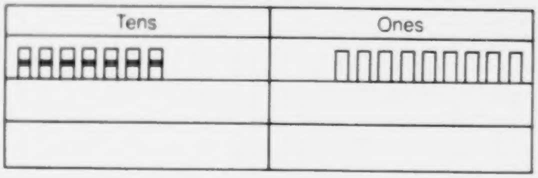
- Have a child show the number of boxes at the beginning of the day in a place-value pocket chart.



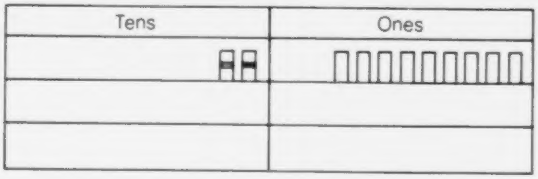
- Discuss the fact that there are not enough ones in the chart to take away 3. "What can we do to get enough ones?" (Regroup 1 of the tens as 10 ones.) Have a child do this.



- Remove 3 of the ones markers. "How many ones are left?"



"We still have to take away 5 of the tens bundles. Will you do that, Juan? How many tens bundles are left?"



"How many boxes of apples were left at the end of the day?"

Repeat with other similar problems.

Variations: Children can use individual sets of beansticks, Cuisenaire rods, Unifix cubes, or base-ten blocks.

After the regrouping process has been made clear, children should be introduced to the algorithm. Again it is important to relate each step in the algorithm to each step with the place-value model. These are the steps for the apple problem in Activity 9-23:

$$\begin{array}{r} 82 \\ -53 \\ \hline \end{array} \qquad \begin{array}{r} \overset{7}{\cancel{8}}{}^{1}2 \\ -53 \\ \hline \end{array} \qquad \begin{array}{r} \overset{7}{\cancel{8}}{}^{1}2 \\ -53 \\ \hline 29 \end{array}$$

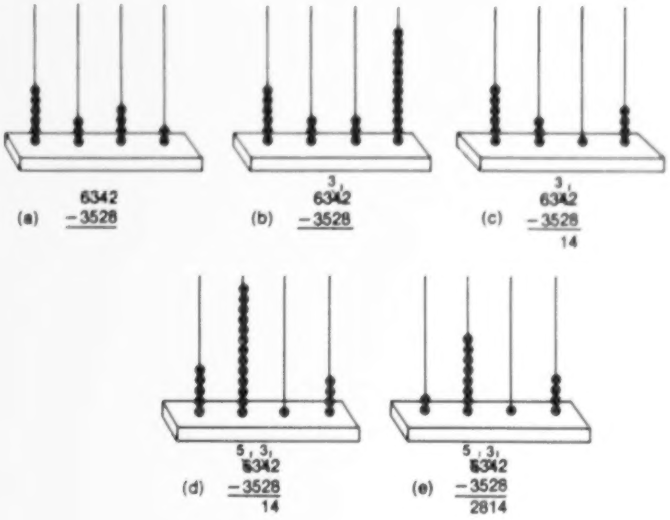
First the original problem is shown. Next the regrouping of 1 ten as 10 ones is shown by the marked-out 8, the small 7, and the 12 ones. The work is then completed with subtraction of 3 from 12 and subtraction of 5 from 7. The line through the 8, the small 7, and the 1 beside the 2 are memory aids that help children keep track of each step as they compute. Children can be encouraged to discontinue using these aids whenever dropping them will not interfere with their ability to compute accurately.

$$\begin{array}{r} 6342 \\ -3528 \\ \hline \end{array}$$

Once children understand the decomposition process with the algorithm for numbers with tens and hundreds, they should be able to subtract with still larger numbers. An abacus with 20 beads on each rod can be used to review the process with larger numbers for children in the middle grades. The process is illustrated in Figure 9-15 for the example in the margin. Only

FIGURE 9-15

Abacus and algorithm showing decomposition for the subtraction sentence 6342 − 3528 = 2814



the minuend, 6342, is shown on the abacus in (a). The steps taken to show decomposition on an abacus are comparable to those used with a place-value pocket chart.

In the middle grades, subtraction involving 0 in the minuend should receive special attention, because problems such as

$$\begin{array}{r} 306 \\ -148 \\ \hline \end{array} \quad \text{and} \quad \begin{array}{r} 6003 \\ -4298 \\ \hline \end{array}$$

are difficult for some children. There are two ways decomposition can be done with examples of this type. To subtract 148 from 306, the thinking might be: "There are no tens, so go to the hundreds. Regroup 1 of the hundreds as 10 tens, leaving 2 hundreds. Then regroup 1 of the tens as 10 ones, giving a total of 16 ones and 9 tens." The subtraction can then be done. The same procedure would be used for the number 6003, except that more regroupings would be required.

$$\begin{array}{r} {}^{2}\cancel{3}\;{}^{9}\cancel{0}\;{}^{1}6 \\ -1\;4\;8 \\ \hline 1\;5\;8 \end{array}$$

$$\begin{array}{r} {}^{5}\cancel{6}\;{}^{9}\cancel{0}\;{}^{9}\cancel{0}\;{}^{1}3 \\ -4\;2\;9\;8 \\ \hline 1\;7\;0\;5 \end{array}$$

Children who have a good understanding of numbers can use the method that is described in the teaching vignette in Chapter 1 (page 13). First, they estimate the remainder—300 − 150 = 150. Next they consider 306 as 30 tens and 6 ones. When the regrouping is done, 1 ten is taken from the 30 tens, leaving 29 tens. The 1 ten is renamed as 10 ones and is added to the 6 ones in the ones place. In one step, all of the required regrouping is completed. If the regrouping is indicated in the algorithm, it appears as shown in the margin. To subtract 4298 from 6003, the regrouping is also done in one step. The number 6003 is thought of as 600 tens and 3 ones.

When 1 ten is renamed as 10 ones, there are 599 tens left. The algorithm in the margin shows how the regrouping might be indicated. By the time children are in the fifth or sixth grade, they should be able to subtract using zeros in this more mature way.

The Equal-Additions Method

The equal-additions method of subtraction is based on the mathematical concept that there is an infinite number of equivalent subtraction problems for a given remainder. Since the answer to both $68 - 14 = \square$ and $70 - 16 = \square$ is 54, the 2 sentences are equivalent expressions for 54. The strength of the equal-additions method of subtraction is the ease and speed with which computation can be done once the process has been mastered. The weakness of this method is that many children find it difficult to understand.

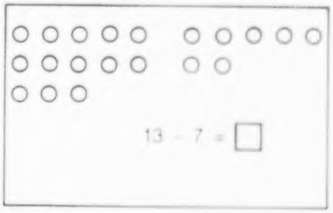
If the equal-additions method is taught, it should be preceded by activities to show that the same number can be added to both subtrahend and minuend without changing their difference.

ACTIVITY 9-24

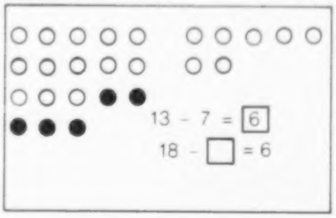
Equal Additions

Use a flannel or magnetic board and markers. Show 2 sets that are to be compared, such as 13 and 7.

Write the sentence $13 - 7 = \square$, and determine the answer.

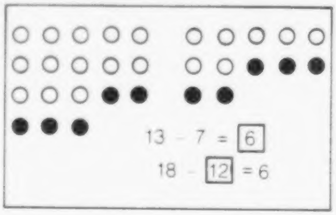


Put 5 more markers with the larger set, making it a set of 18.



Ask, "What must we do to the second set so that the difference remains the same?" Help the children realize that 5 markers must also be put with the second set.

Complete the second sentence, 18 − 12 = 6.

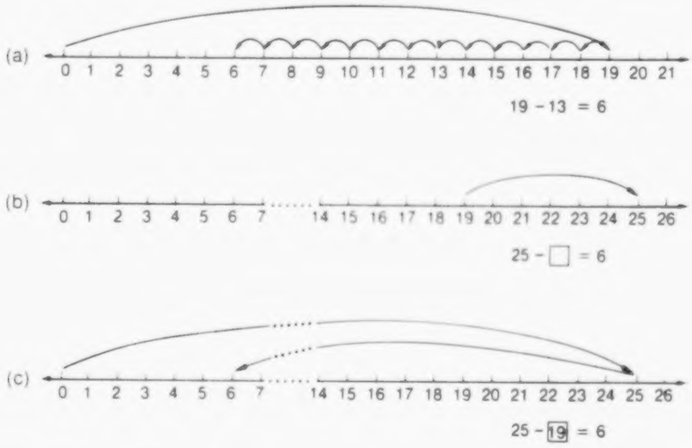


Repeat with other sets.

The number line is another useful aid for discovering this principle, as Figure 9-16 shows. Write a subtraction sentence, such as 19 − 13 = ☐, and represent the minuend on the number line with an arrow extending from 0 to 19 and the subtrahend by an arrow extending from 19 backward 13 steps to the 6, as in (a). Then increase the minuend by a given amount, say 6, so it becomes 25. Show the new minuend on the number line (b). The children can then discuss what must be done to the subtrahend so the difference, 6, will remain unchanged. An arrow drawn from the new minuend and extending backward to 6 on the line will show the new subtrahend. A look at this arrow reveals that the new subtrahend is the number of the old subtrahend increased by the same amount as was the original minuend (c).

FIGURE 9-16

Number line and sentences showing that the same number must be added to both minuend and subtrahend if the difference is to remain unchanged


(a)
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21
19 − 13 = 6
(b)
0 1 2 3 4 5 6 7 14 15 16 17 18 19 20 21 22 23 24 25 26
25 − ☐ = 6
(c)
0 1 2 3 4 5 6 7 14 15 16 17 18 19 20 21 22 23 24 25 26
25 − 19 = 6

In addition to understanding that the difference remains the same if a given number is added to both minuend and subtrahend, an understanding of place value is also essential before the equal-additions method of subtracting should be introduced. Once an adequate background for the process has been established, proceed as follows. Write a subtraction example on the chalkboard:

$$\begin{array}{r} 632 \\ -216 \\ \hline \end{array}$$

$$\begin{array}{r} 6\;3\;{}^{1}2 \\ -2\;{}^{2}1\;6 \\ \hline 4\;1\;6 \end{array}$$

$$\begin{array}{r} 3\;{}^{1}2\;4\;{}^{1}6 \\ -{}^{2}1\;9\;{}^{4}3\;7 \\ \hline 1\;3\;0\;9 \end{array}$$

Children see that it is not possible to subtract 6 ones from 2 ones. The subtraction is made possible when 10 ones are added to the 2 ones in the minuend to make 12 ones. The children then subtract 6 from 12 and record a 6 in the ones place of the algorithm. Call the children's attention to the fact that because 10 was added to the minuend, 10 must also be added to the subtrahend. This is done by adding 1 ten to the number in the tens place of the subtrahend. Then 2 tens are subtracted from 3 tens, and the answer is recorded in the tens place. The subtraction is completed by subtracting 2 hundreds from the 6 hundreds. The completed algorithm would appear as shown above in the margin. The sentence 3246 − 1937 = ☐ is shown in the algorithm in the margin. Once children learn the process, the thinking is 16 − 7 = 9, 4 − 4 = 0, 12 − 9 = 3, and 3 − 2 = 1. When the number is in a place-value position in the minuend is smaller than the number in the same place-value position of the subtrahend, it is automatically increased by 10, and the subtraction for that position is completed. Each time this is done, the number in the subtrahend in the place-value position to its immediate left is increased by 1. The process becomes automatic when it is well understood and sufficiently practiced. This process presents no particular difficulties with zeros in the minuend.

Estimating Remainders

There are never more than 2 numbers involved in subtraction. For that reason, rounding-off is the most used estimation process. Numbers of any size can be rounded to an appropriate place value position and the estimate completed.

USING COMPUTERS AND CALCULATORS

The computer and calculator can be used to improve children's understanding of operations, applications of addition and subtraction in real-life situations, and explorations of interesting mathematical ideas. Do not use software as a substitute for concrete materials and interaction between you and children; use programs with individual children who have demonstrated understanding and know many facts. The programs you use will be more interesting to children who feel successful with them. A chart that shows improvement or completion of lessons helps children follow their progress. Many addition and subtraction practice programs are available for elementary mathematics. Many textbook series, in fact, offer a complete software package for practice. Many other software publishers also have addition/subtraction packages (see Appendix B).

Two programs previously mentioned in this chapter for basic fact practice with addition and subtraction also contain more advanced exercises. *Stickybear Math* progresses through 20 levels to 4-digit operations with regrouping. *Math Blaster* offers a full range of practice problems. *Estimate*[16] is a timed-practice program aimed at improving estimation of answers to problems in addition, subtraction, multiplication, division, and percent. A computational practice program that challenges students is *Quations*,[17] played on a grid similar to a Scrabble™ board. Students fill in the blanks using numbers and all the operations to score points. Much strategy is needed to place numbers in the squares and maximize the scores.

Many children can also write their own drill programs. A prototype Logo program that randomly generates addition problems is easy to modify for other operations:

```
TO DRILL
MAKE "ADDEND1 RANDOM 50
MAKE "ADDEND2 RANDOM 50
PRINT (SENTENCE :ADDEND1 [+] :ADDEND2 [=])
WAIT 50
DRILL
END
```

This simple procedure is easy to adapt to other problems by changing the number with RANDOM. It can also be speeded up or slowed down, or the sign can be changed. Children may want to improve the program so that it checks the answer or keeps score. More elaborate math quiz programs can be found in many Logo books.[18]

Calculator activities also emphasize addition and subtraction. One activity that extends place-value understanding and gives practice with addition is played with 2 or 3 four-digit numbers created by rolling a die or with a spinner, as in Game 9-2. Game 9-3 provides practice with estimation and addition. Game 9-4 has been played in many ways with a variety of materials, but the calculator offers a new means of playing it.

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[16] Minnesota Educational Computing Consortium, 3490 Lexington Ave., North, St. Paul, MN 55126.

[17] Scholastic Inc., PO Box 7502, Jefferson City, MO 65102.

[18] See Dan Watt, *Learning with Logo* (New York: BYTE/McGraw-Hill, 1983), pp. 198–200; or Steve Tipps and Glen Bull, *Beginning with Logo* (Englewood Cliffs, N.J.: Prentice-Hall, 1987), pp. 97–101.

GAME 9-2

Roll and Add

- Materials: calculator, die.
- Two to four children play. Each marks 3 lines with places for 4 numbers in each line.

```
   ___ ___ ___ ___
   ___ ___ ___ ___
 + ___ ___ ___ ___
 _________________
```

- Roll the die 12 times. As a number is determined, each student places it in a blank. The objective is to create the largest possible sum.
- Use the calculator to determine the sum.
- Variations:
  (a) Play the game with a spinner with digits 0–9.
  (b) Play the game with the objective of creating the smallest possible sum.
  (c) Use just 2 four-digit numbers; students try for the largest or the smallest difference.

GAME 9-3

Bull's-eye

- Create a pack of 20 index cards with numerals on each card. A first set may contain numerals for numbers between 10 and 100; a second set between 100 and 1000; a third set between 1000 and 10,000. Use the numerals according to the ability of your students.
- Three or four children play the game. Each child draws 3 or 4 cards and estimates the sum.
- Write down the estimate. With the calculator, find the actual sum.
- Use the calculator to find the difference between the estimate and the actual sum. The difference between the actual sum and the estimate becomes a score.
- Play 4 rounds and sum the 4 scores. The lowest total wins.

- A range of scores appropriate for the cards is displayed on a chart. For 2-digit addition, the center of the bull's-eye might be 0–10, the first ring for scores 11–30, the next ring for scores 31–60, and the last ring for scores 61–100. Put colored markers on a bull's-eye chart to show how close each child's estimate was to the actual sum. Markers near the bullseye indicate which children's estimates are most accurate.
- Variation: Estimation can also be done with the differences between 2 drawn cards.

GAME 9-4

NIM

- Select a target number, say 50.
- The first player enters a 1-digit number into the calculator.
- The second player adds a second 1-digit number to the first.
- Players alternate adding 1-digit numbers until 1 player reaches exactly 50 and is declared winner of the round.
- Encourage children to work out a strategy for winning. Suggest that they try their strategies with smaller target numbers.
- Variations: Start at 30 and subtract 1-digit numbers less than 5 to reach 0.

Other calculator activities include work with a subtraction investigation, palindromic numbers, and magic squares.

Subtracting to 495

This activity is based on the fact that a series of subtractions involving the largest and smallest numbers created from 3 digits, with at least 1 digit different from the other 2, will eventually yield the number 495. Try the digits 8, 6, and 4. The largest 3-digit number is 864, and the smallest is 468. Subtract 468 from 864: 864 − 468 = 396; subtract 369 from 963: 963 − 369 = 594; subtract 459 from 954: 954 − 459 = 495. Try the digits 9, 8, and 2: 982 − 289 = 693; 963 − 369 = 594; 954 − 459 = 495. Try 8, 9, and 9: 998 − 899 = 99; 990 − 099 = 891; 981 − 189 = 792; 972 − 279 = 693; 963 − 369 = 594; 954 − 459 = 495. Note that zeros are used to make 99 a 3-digit number. This practice is followed any time the difference is less than 100. Encourage your children to test 2- and 4-digit numbers to see if a similar result occurs.

Palindromic Numbers

$$\begin{array}{rr} 42 & 421 \\ +24 & +124 \\ \hline 66 & 545 \end{array}$$

A palindrome is a number that reads the same backward as forward, such as 232, 46,564, 1,234,321. An interesting investigation involves addition to generate palindromes. Select any whole number. Add that number to the one you get when you reverse its digits. If the sum is a palindrome, the addition is complete, as in the examples in the margin. If the first addition does not give a palindrome, use the sum and its reversed digits as new addends. Add these numbers. If this sum is not a palindrome, repeat the process until a palindrome appears. Note that palindromes can be generated from numbers smaller than 10.

$$\begin{array}{rrr} 69 & 596 & 3 \\ +96 & +695 & +3 \\ \hline 165 & 1291 & 6 \\ 561 & 1921 & 6 \\ \hline 726 & 3212 & 12 \\ 627 & 2123 & 21 \\ \hline 1353 & 5335 & 33 \\ 3531 & & \\ \cline{1-1} 4884 & & \end{array}$$

Interested children will test many numbers. Often the numbers they use will exceed the machine's capacity, so they may revert to paper-and-pencil computation or attempt to devise ways to do the work on the calculator.

Magic Squares

Magic squares have a long history. Their mystery has interested professional and amateur mathematicians for centuries. Until now, children who have been interested in magic squares have found the computation required in connection with them so tedious that they do not extend their investigations beyond the more simple squares. The calculator provides the means for taking the burden out of the computing, so interested students can investigate much more sophisticated squares. Magic squares with 3 × 3, 4 × 4, and 5 × 5 cells are illustrated in Figure 9-17. Add the numbers in each column, row, and diagonal. If the sum is the same for each group of numbers, you have a magic square. Children should be encouraged to work with these squares and then to generate their own if possible.

FIGURE 9-17

Magic squares

8	1	6
3	5	7
4	9	2

16	2	3	13
5	11	10	8
9	7	6	12
4	14	15	1

17	24	1	8	15
23	5	7	14	16
4	6	13	20	22
10	12	19	21	3
11	18	25	2	9

SPECIAL PROBLEMS WITH ADDITION AND SUBTRACTION

While most students have little difficulty understanding the meanings of addition and subtraction, learning the facts, and performing computations effectively, some children do have difficulty in these areas. The following examples illustrate types of errors found in some students' work:

1. Addition
   a. Adding the numbers' digits as though each digit were a separate number:

$$\begin{array}{r} 32 \\ +65 \\ \hline 16 \end{array} \qquad \begin{array}{r} 94 \\ +27 \\ \hline 22 \end{array} \qquad \begin{array}{r} 87 \\ +49 \\ \hline 28 \end{array}$$

   b. Beginning at the left:

$$\begin{array}{r} 3^{2}4^{3}2 \\ +9\ 7\ 2 \\ \hline 1\ 1\ 7 \end{array} \qquad \begin{array}{r} 4\ 6^{3}9 \\ +2\ 7\ 5 \\ \hline 6\ 1\ 17 \end{array} \qquad \begin{array}{r} 3^{1}4\ 2\ 9 \\ +8\ 1\ 5\ 6 \\ \hline 1\ 6{,}7\ 15 \end{array}$$

   c. Failing to regroup:

$$\begin{array}{r} 342 \\ +972 \\ \hline 12{,}114 \end{array} \qquad \begin{array}{r} 439 \\ +275 \\ \hline 61{,}014 \end{array} \qquad \begin{array}{r} 5798 \\ +6204 \\ \hline 119{,}912 \end{array}$$

2. Subtraction
   a. Beginning at the left:

$$\begin{array}{r} \cancel{4}{}^{1}7 \\ -2\,9 \\ \hline 2\,6 \end{array} \qquad \begin{array}{r} 7^{1}\cancel{3}^{1}6 \\ -5\,8\,7 \\ \hline 2\,5\,9 \end{array}$$

   b. Subtracting the smaller number in a column from the larger, regardless of where it is in the algorithm:

$$\begin{array}{r} 47 \\ -29 \\ \hline 22 \end{array} \qquad \begin{array}{r} 736 \\ -587 \\ \hline 251 \end{array}$$

Children who commit errors such as these demonstrate that they do not fully understand the procedures for adding or subtracting numbers. While the diagnostic-developmental teaching mode recommended in this book will minimize errors, you may encounter some children who commit them in their work. When you do, immediate attention is necessary. If errors are caught early, overcoming them is easier than when they go undetected and become habitual. An interview is an appropriate way to assess the child's faulty thinking and to determine which materials and procedures are appropriate for clearing up misunderstandings. The errors may be eliminated simply through discussion with the child or may require reteaching with concrete materials.

You may encounter some children for whom the standard modes of computing are ineffective. Children who have severe problems with addition and subtraction may benefit from special procedures that have been developed for them. The activities described here employ visual, auditory, and kinesthetic modes of presenting facts and algorithms and have children respond to them.

In *Oregon Math*, children learn verbal and visual cues that direct their actions and learn to add and subtract by counting on their fingers. For example, when addition is done, a lesson proceeds this way:

1. $\begin{array}{r} 3 \\ +2 \\ \hline \end{array} \quad \begin{array}{r} 6 \\ +2 \\ \hline \end{array} \quad \begin{array}{r} 8 \\ +1 \\ \hline \end{array} \quad \begin{array}{r} 5 \\ +2 \\ \hline \end{array}$

2. TODAY WE ARE GOING TO ADD.
   Point to the + sign and say:
   THIS SIGN SAYS PLUS. IT TELLS US TO ADD.
3. THE BOTTOM NUMBER TELLS US HOW MANY FINGERS TO PUT UP.
   Point to the number 2 in the first problem.

4. HOW MANY FINGERS DO WE PUT UP?
   Tap the 2. Students respond in unison, "2." (If students do not respond in unison, say WAIT FOR THE SIGNAL.)
5. SHOW ME.
   Put up 2 fingers as students each put up 2 fingers.
6. THE TOP NUMBER TELLS US WHERE TO START COUNTING.
   Point to the number 3 in the first problem.
7. WHERE DO WE START COUNTING?
   Tap the 3 as students respond in unison, "3."
8. *Tap* the 3 on the board, saying:

   3, ...

   With students, touch 1 finger and put it down, saying:

   4

   And then the other finger, saying:

   5

9. ARE ALL THE FINGERS GONE?
   Tap. Students respond in unison, "Yes."
10. WHERE DID WE STOP COUNTING?
    Tap. Students respond in unison, "5."
11. WHAT IS THE ANSWER?
    Tap. Students respond in unison, "5."
12. $\boxed{5}$
13. Repeat steps 3–12 for remaining problems.[19]

Although *Oregon Math* is designed for students with severe learning problems, it has features that can be adapted for students in any classroom who experience problems learning basic facts.

Terence Kramer and David A. Krug[20] discuss a different procedure they have used with educable mentally retarded students. It uses fixed reference points on each of the numerals. Students look at a numeral and imagine the

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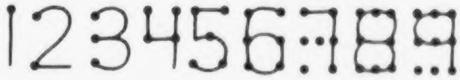
[19] Lynn C. Weigel and Gerald A. Weigel, *Oregon Math Computation*, Speciman Set (Tigard, Ore.: C. C. Publications, 1979), p. 1. (Used by permission.)

[20] Terence Kramer and David A. Krug, "A Rationale and Procedure for Teaching Addition," *Education and Training of the Mentally Retarded*, VIII, No. 3 (October 1973), pp. 140–145.

FIGURE 9-18

Reference points for numerals 1 through 9

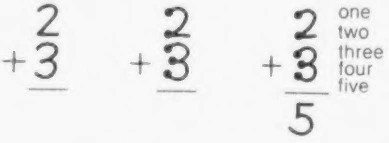


points; then they touch or make small marks and count the points. Figure 9-18 shows each of the numerals with its reference points.

Once children understand the pattern for each number, they are ready to use them for adding. Figure 9-19 shows the sequence of steps. First the pattern is put on each addend; then they are counted and the sum is recorded.

FIGURE 9-19

Sequence of steps for adding with reference points

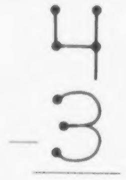


A second way of adding is for the child to say the first addend—"2"—and then count on the second addend—"3, 4, 5."

Subtraction is done as shown in Figure 9-20. The child marks the points for the minuend. One point is marked out for each point in the subtrahend. The points that remain unmarked are the difference between the 2 numbers.

FIGURE 9-20

Subtraction with marked-out reference points



A procedure similar to this one has been extended to also include multiplication and division and is sold under the name *Touch Math*™.[21]

Special algorithms have been developed for students in grade six and beyond who cannot master one of the conventional algorithms. These low-stress algorithms do not reduce the number of steps when adding or subtracting. However, they do change the order in which the steps are completed so that the student completes all of 1 process before moving on to the next. Barton Hutchings, the originator of the algorithms, contends that they can be mastered readily after a brief training period, are performed with less stress than conventional algorithms, and give the learner great computational power in a short time.[22]

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[21] Touch Learning Concepts ™, Inc., 104 Manitou Avenue, Manitou Springs, CO 80829.

[22] Barton Hutchings, "Low-Stress Algorithms," in *Measurement in School Mathematics*, 1976 Yearbook (Reston, Va.: National Council of Teachers of Mathematics, 1976), p. 219.

Addition Algorithm

$$\begin{array}{r} 5\\ +7\\ \hline 12 \end{array} \qquad \begin{array}{r} 5\\ +{}_{1}7_{2} \end{array}$$

$$\begin{array}{r} 6\\ +2\\ \hline 8 \end{array} \qquad \begin{array}{r} 6\\ +2_{8} \end{array}$$

The addition algorithm is similar to the conventional algorithm, but it has a slightly different form of notation, called half-space notation. The difference is shown in the margin, where the conventional notation for a basic fact is shown on the left, and half-space notation for the same fact is on the right. For a combination having a sum less than 10, the single numeral is written slightly below and to the right of the 2 addends. In half-space notation, the line separating the addends from their sum is omitted in basic facts. The reason for this notation will become clear as the description continues.

$$\begin{array}{r} 8\\ {}_{1}4_{2}\\ 6_{8}\\ {}_{1}9_{7}\\ +{}_{1}8_{5}\\ \hline 35 \end{array}$$

The low-stress algorithm is particularly useful for adding when there are 3 or more addends. The example in the margin shows that all addition is recorded as the work is completed. Each fact in the set of addends has its sum recorded; no mental accumulation of sums is necessary. In the example, the process is this: add 8 + 4, record 12; add 2 + 6, record 8; add 8 + 9, record 17; add 7 + 8, record 15. Now all addends have been added, so record 5 below the line. Count the number of times a ten was recorded at the left of the column; record 3 below the line. The sum for this addition is 35.

$$\begin{array}{rr} ② & \\ 2_{4} & 6\\ {}_{1}8_{2} & {}_{1}4_{0}\\ 7_{9} & 8_{8}\\ {}_{1}4_{3} & {}_{1}9_{7}\\ \hline 2\,3 & 7 \end{array}$$

Addition with addends greater than 10 is illustrated in the margin. Add 6 + 4, 10; 0 + 8, 8; 8 + 9, 17; record 7 below the line. Count the number of times ten was recorded, then "carry" 2 to the tens column. Add 2 + 2, 4; 4 + 8, 12; 2 + 7, 9; 9 + 4, 13. Record 3 below the line. Count the tens (these are actually hundreds since the addition is in the tens column), then record 2 below the line. The answer is 237.

A student who has mastered the process with numbers smaller than 10, and then smaller than 100, can quickly move to addition with quite large numbers and several addends. For an older student who has experienced failure with the conventional algorithm, success with the low-stress computation will perhaps bolster the child's self-esteem and help change a negative attitude toward mathematics to a positive one.

Subtraction Algorithm

When doing subtraction, the student completes all regrouping first. The following example illustrates the process. The original combination is shown in (a). The regrouping is placed between the minuend and subtrahend, as shown in the series of steps in (b), (c), and (d).

$$\text{(a)}\ \begin{array}{r} 6241\\ -3675\\ \hline \end{array} \quad \text{(b)}\ \begin{array}{r} 6\,2\,4\,1\\ 3^{1}1\\ -3\,6\,7\,5\\ \hline \end{array} \quad \text{(c)}\ \begin{array}{r} 6\,2\,4\,1\\ 1^{1}3^{1}1\\ -3\,6\,7\,5\\ \hline \end{array} \quad \text{(d)}\ \begin{array}{r} 6\,2\,4\,1\\ 5^{1}1^{1}3^{1}1\\ -3\,6\,7\,5\\ \hline \end{array} \quad \text{(e)}\ \begin{array}{r} 6\,2\,4\,1\\ 5^{1}1^{1}3^{1}1\\ -3\,6\,7\,5\\ \hline 2\,5\,6\,6 \end{array}$$

Finally the subtraction is completed and the answer is recorded. Subtraction with zeros in the minuend is done this way:

$$\text{(a)}\ \begin{array}{r} 4002 \\ -2146 \\ \hline \end{array} \qquad \text{(b)}\ \begin{array}{r} 4\,0\,0\,2 \\ 3\quad\ {}^{1}2 \\ -2\,1\,4\,6 \\ \hline \end{array} \qquad \text{(c)}\ \begin{array}{r} 4\,0\,0\,2 \\ 3\,9\,9{}^{1}2 \\ -2\,1\,4\,6 \\ \hline \end{array} \qquad \text{(d)}\ \begin{array}{r} 4\,0\,0\,2 \\ 3\,9\,9{}^{1}2 \\ -2\,1\,4\,6 \\ \hline 1\,8\,5\,6 \end{array}$$

The original example is shown in (a). In (b) the ones are regrouped, and the 4 in the thousands place is changed to 3, with the zeros being ignored at this point. After this regrouping is completed, each 0 is replaced by a 9, as in (c). Now the subtraction can be completed (d). The process with a larger minuend having zeros is illustrated in the following example, with the original problem shown in (a). In (b) the regrouping of numbers greater than 0 is completed. In (c) each 0 is replaced by a 9; the answer is shown in (d).

$$\text{(a)}\ \begin{array}{r} 430002 \\ -246483 \\ \hline \end{array} \qquad \text{(b)}\ \begin{array}{r} 4\,3\,0\,0\,0\,2 \\ 3{}^{1}2\qquad\ {}^{1}2 \\ -2\,4\,6\,4\,8\,3 \\ \hline \end{array} \qquad \text{(c)}\ \begin{array}{r} 4\,3\,0\,0\,0\,2 \\ 3{}^{1}2\,9\,9{}^{1}2 \\ -2\,4\,6\,4\,8\,3 \\ \hline \end{array} \qquad \text{(d)}\ \begin{array}{r} 4\,3\,0\,0\,0\,2 \\ 3{}^{1}2\,9\,9\,9{}^{1}2 \\ -2\,4\,6\,4\,8\,3 \\ \hline 1\,8\,3\,5\,1\,9 \end{array}$$

The primary advantage of this algorithm for the student is that all of one part of the work, the regrouping, is completed before the next part, the subtraction, is done. This reduces the cognitive load and the possibility of error, because the child does not shift from thinking about regrouping to subtracting, back to regrouping, then subtracting again, and so on. An advantage of both algorithms for the teacher is that it is easy to detect errors. By examining a child's work, one can spot errors with basic addition and subtraction facts, failure to "carry" in addition, and regrouping errors in subtraction.

Summary

Addition and subtraction are usually introduced in the first grade; and once each is introduced, they are usually studied simultaneously. There are 100 basic addition facts, which are all of the possible combinations of pairs or addends smaller than 10 and their sums. Each addition combination has a corresponding subtraction fact. The real-life situation that gives rise to addition is the joining of 2 or more sets having no members in common; there are 4 situations that give rise to subtraction.

Children develop readiness for addition and subtraction as they learn to count using concrete materials. Working in the concrete-manipulative mode with real objects and markers gives children opportunities to join, separate, and compare the sizes of sets of objects; this is the basis for understanding the meaning of addition and subtraction and the use of abstract sentences.

Once children know the meaning of addition and subtraction, they need to use strategies for learning the facts and to engage in activities that will help them remember each fact. Strategies that use families of facts, doubles and near doubles, adding 1 less than 10, and the addition table should be used. Computer programs, calculator activities, and games as well as practice pages in workbooks provide a variety of practice experiences. Children's understanding of the commutative and associative properties and the identity element for addition should be developed through work with manipulative materials.

When children have a good understanding of place value for numbers through 99 and of the processes of addition and subtraction, they are ready for addition and subtraction with sums up to 99. Later, addition and subtraction are extended to whole numbers of any size. Skill in adding by endings is important so that children can do certain addition, as when adding columns of numbers and doing some types of multiplication, without paper and pencil. Addition of larger numbers will involve some situations where no regrouping is required and some where it is. Place-value pocket charts, beansticks, an abacus, and other models will make this addition meaningful to children. Rounding-off, compatible number, and front-end strategies provide means for children to estimate answers.

Subtraction of larger numbers can be performed by decomposition, equal addition, and other processes. Decomposition is generally favored because its meaning can be illustrated with place-value models. Equal addition is performed rapidly by persons who learn to do it. Eventually children learn to use the addition and subtraction algorithms without benefit of manipulative aids. Rounding off is most often used for estimating subtraction answers.

The calculator and computer are useful for engaging children in interesting and instructive learning activities and investigations dealing with addition and subtraction. Children should also have opportunities to use the calculator for routine computation with these operations in problem-solving situations.

Many drill-and-practice programs are available for students. Teachers should, however, be careful that students are well prepared with concrete materials and concepts before beginning drills. When students are prepared, they often enjoy success with the drill programs. Software should be seen as a supplement rather than a substitute for thorough instruction emphasizing understanding of operations. With careful review, teachers may also find programs that are more creative and enhance understanding of mathematical operations.

Special procedures for dealing with basic facts and special algorithms have been developed for children with special learning problems. The fact activities combine visual, auditory, and kinesthetic experiences to help children learn to deal with facts. These special algorithms are designed to lessen stress by separating steps during computation.

Study Questions and Activities

1. Addition and subtraction are inverse operations, meaning that subtraction "undoes" addition. List 5 everyday, nonmathematical situations illustrating inverse operations that are good examples to use with children. For example, opening and closing a door are inverse operations.
2. Subtraction seems difficult for many children. Often they are only introduced to the "take-away" meaning for subtraction. Why is use of this meaning alone unsatisfactory for students? Write a story situation that illustrates each of the 4 subtraction situations described in this chapter. Draw a picture to illustrate each one. Some children are taught to read the minus sign (−) as "take-away." What do you think of this practice? What difficulties might arise from its exclusive use?
3. Obtain a practice page of 100 addition or subtraction facts and take a 1-minute timed test on them. How many did you get right? How did you feel at the end of the test? Discuss with fellow students any memories you have of timed tests in elementary school. What are the possible harmful aspects? What ways can you think of to overcome the pressure that many children feel from timed tests?
4. Think of everyday situations that illustrate the commutative property. Whether you put cereal in the bowl first or milk first makes no difference. Opening the door before walking in or out, however, is much different from walking in or out before opening. Order matters, so this situation is not commutative. List 5 more everyday commutative situations and 5 situations that are not commutative.
5. The method of reference points on numerals is being used in some school districts in a supplementary mathematics program called Touch Math™. The method is, however, not without controversy. Read "Beware of Tapping Pencils" by Roberta Flexer and Naomi Rosenberger and "Encourage Tapping Pencils" by Jan Bullock and Joanna Love (see For Further Reading section). After reading, summarize the possible strengths of this method and some potential problems. Discuss these with a teacher who has used the method, and get a reaction.
6. Use the *Arithmetic Teacher* and other resources to find addition and subtraction activities for your teaching file. The 1978 Yearbook of the National Council of Teachers of Mathematics, entitled *Developing Computational Skills*, contains many suitable activities. If possible, locate at least 1 game or activity for each of these purposes: improving understanding of addition or subtraction, learning combinations, practicing basic facts, addition or subtraction without regrouping, addition or subtraction with regrouping.
7. Use a place-value pocket chart, beansticks, and a classroom abacus to

practice the following examples until you are proficient at demonstrating the process:

(a) $\begin{array}{r}23\\+41\\\hline\end{array}$ (b) $\begin{array}{r}68\\+26\\\hline\end{array}$ (c) $\begin{array}{r}203\\+429\\\hline\end{array}$ (d) $\begin{array}{r}688\\+247\\\hline\end{array}$

8. Use both the decomposition and equal-additions method of subtraction to solve the following problems:

(a) $\begin{array}{r}62\\-48\\\hline\end{array}$ (b) $\begin{array}{r}436\\-209\\\hline\end{array}$ (c) $\begin{array}{r}943\\-387\\\hline\end{array}$ (d) $\begin{array}{r}4003\\-3829\\\hline\end{array}$

With which procedure can you subtract more rapidly? Why do you think that is so? Use a place-value pocket chart, beansticks, or chips to represent the decomposition process.

9. Sherrill (see For Further Reading section) compared performance of children in the third-grade class who learned the decomposition method of subtraction with another group who learned equal additions. What were the results of his study, and what implications do you draw from his report of these children's performances?

10. The article by Beede (see For Further Reading section) describes a game called SAM (Speed and Accuracy in Mathematics). He shows how to play the game for practice with division problems. Prepare this game using appropriate addition and subtraction examples for grades four to six.

11. Estimate the sums of the following problems. Make a mental note of the process you use to arrive at your answer. Is your strategy rounding-off, using compatible numbers, or front-end estimation?

(a) $\begin{array}{r}450\\923\\128\\554\\721\\+218\\\hline\end{array}$ (b) $\begin{array}{r}3789\\472\\2091\\8357\\1826\\+6649\\\hline\end{array}$ (c) $\begin{array}{r}711\\826\\138\\624\\448\\+437\\\hline\end{array}$

Did you use the same strategy in each problem? Ask several people you know to estimate the sums, and ask them how they did it. They may not know the name of the strategy but will be able to describe their method. Ask several fifth- or sixth-grade students to estimate the sum and describe their method to you. Can you draw any conclusions from this activity about estimation strategies?

For Further Reading

Ashlock, Robert B., and Carolynn A. Washbon. "Games: Practice Activities for the Basic Facts," in *Developing Computational Skills*, 1978 Yearbook, National Council of Teachers of Mathematics (Reston, Va.: The Council), pp. 39–50. Describes the role of practice activities and guidelines for selecting and preparing them, along with examples of games. A short list of steps for creating games helps the teacher design his or her own games.

Backman, Carl A. "Analyzing Children's Work Procedures," in *Developing Computational Skills*, 1978 Yearbook, National Council of Teachers of Mathematics (Reston, Va.: The Council), pp. 177–195. Diagnosis requires more than analysis of tests. Children's daily work—both results and processes—needs to be analyzed too. Suggestions for doing this and for correcting errors in computation are explained and illustrated.

Baroody, Arthur J. "Children's Difficulties in Subtraction: Some Causes and Cures," *Arithmetic Teacher*, XXXII, No. 3 (November 1984), pp. 14–19. Counting strategies provide the basis for understanding subtraction facts.

Beardslee, Edward C. "Teaching Computational Skills with a Calculator," in *Developing Computational Skills*, 1978 Yearbook, National Council of Teachers of Mathematics (Reston, Va.: The Council), pp. 226–241. Explains a variety of activities with a calculator dealing with counting, addition and subtraction, multiplication and division, problem solving, pattern investigations, and decimals.

Beattie, Ian D. "Modeling Operations and Algorithms," *Arithmetic Teacher*, XXXIII, No. 6 (February 1986), pp. 23–28. Describes rationale and materials needed to demonstrate addition and subtraction algorithms.

Beede, Rudy B. "Speed and Accuracy in Mathematics," *Arithmetic Teacher*, XXVII, No. 5 (January 1980), pp. 44–45. SAM is a competitive game for 2 to 4 children for practicing operations with algorithms. Materials described in the article deal with division, but the game is easily adapted for any operation.

Bullock, Jan, and Joanna Love. "Encourage Tapping Pencils," *Arithmetic Teacher*, XXXIV, No. 5 (January 1987), pp. 3–4. In this letter to the editor, the founders of *Touch Math*™ defend the use of reference points in computing. They stress that it is a supplement to other methods of instruction that gives children an efficient way to compute until they have memorized facts. They also believe that the method is effective for all children.

Campbell, Patricia F. "What Do Children See in Mathematics Textbook Pictures?" *Arithmetic Teacher*, XXVIII, No. 5 (January 1981), pp. 12–16. The author interviewed first-grade children to get their interpretations of pictures portraying addition and subtraction situations. She found 4 levels of interpretation, ranging from no understanding to full understanding. She concludes that teachers need to know each child's level of understanding.

Flexer, Roberta J., and Naomi Rosenberger. "Beware of Tapping Pencils," *Arithmetic Teacher*, XXXIV, No. 5 (January 1987), pp. 6–10. This article opposes the use of reference points.

Hutchings, Barton. "Low-Stress Algorithms," in *Measurement in the Classroom*, 1976 Yearbook, National Council of Teachers of Mathematics (Reston, Va.: The

Council), pp. 218–239. Explains low-stress algorithms for all 4 operations. The processing of multiplication and division facts using the low-stress addition process are also possible.

———. "Low-Stress Subtraction," *Arithmetic Teacher*, XXII, No. 3 (March 1975), pp. 226–322. Discusses the low-stress algorithm for subtraction. Includes several examples and an extensive bibliography dealing with research into the addition and subtraction algorithms.

Hutcheson, James W., and Carol E. Hutcheson. "Homemade Device for Quick Recall of Basic Facts," *Arithmetic Teacher*, XXV, No. 4 (January 1978), pp. 54–55. An overhead transparency using a grid of random digits and paper shields that reveal rows and columns of digits offers opportunity for several skill-development activities; for example, recall of basic facts (sums and products), comparing numbers, reading 2- and 3-place numbers.

King, Julia A. "Missing Addends: A Case of Reading Comprehension," *Arithmetic Teacher*, XXX, No. 1 (September 1982), pp. 44–45. Children are frequently confused by simple missing addend sentences. The author contends that it is a reading problem, not a mathematics problem. A game, "Are You Kidding," is recommended as one way to help children who have this problem.

Logan, Henrietta L. "Renaming with a Money Model," *Arithmetic Teacher*, XXVI, No. 1 (September 1978), pp. 23–24. Children who have difficulty grasping the meaning of subtraction with regrouping may be aided by using dollars, dimes, and pennies as models for place value. First, different ways of representing a given amount of money are practiced; then the money is used as a model for the subtraction algorithm.

Merseth, Katherine K. "Using Materials and Activities in Teaching Addition and Subtraction Algorithms," in *Developing Computational Skills*, 1978 Yearbook, National Council of Teachers of Mathematics (Reston, Va.: The Council), pp. 61–77. Explains readiness activities with base-ten blocks, record-keeping activities with mats and blocks, and addition and subtraction activities with the mats and blocks. Activities leading to abstractions of the processes are part of the discussion.

Nichol, Margaret. "Addition Through Palindromes," *Arithmetic Teacher*, XXVI, No. 4 (December 1978), pp. 20–21. There are both words and numbers that are palindromes. Of particular interest are the suggestions for using palindromes to provide addition practice. Also of interest is the fact that it takes 24 steps to change 89 to a palindrome.

Rathmell, Edward C. "Using Thinking Strategies to Teach the Basic Facts," in *Developing Computational Skills*, 1978 Yearbook, National Council of Teachers of Mathematics (Reston, Va.: The Council), pp. 13–38. Presents strategies for organizing basic facts for aiding children's understanding and retention. Includes many illustrations.

Rheins, Gladys B., and Joel J. Rheins. "A Comparison of Two Methods of Compound Subtraction," *Arithmetic Teacher*, II, No. 3 (October 1955), pp. 63–69. Reports the results of a study comparing groups of children taught by 2 subtraction methods, the decomposition and equal-additions methods. The conclusion is that the decomposition method is a better way to introduce compound subtraction.

Shaw, Jean. "Punchy Mathematics," *Arithmetic Teacher*, XXXI, No. 2 (October 1983), pp. 10–14. The hole punch and paper provide children with many number activities leading to addition, subtraction, multiplication, and geometric ideas.

Sherrill, James M. "Subtraction: Decomposition versus Equal Addends," *Arithmetic Teacher*, XXVII, No. 1 (September 1979), pp. 16–17. Sherrill compared 2 groups of third-grade children to determine whether the group that learned the decomposition method or the one that learned the equal-additions method of subtraction performed better at the close of the instructional period. His results confirmed those of many others who have compared the 2 methods: children who learned the decomposition method worked with greater accuracy and understanding.

Thompson, Charles S., and William P. Dunlop. "Basic Facts: Do Your Children Understand or Do They Memorize?" *Arithmetic Teacher*, XXV, No. 3 (December 1977), pp. 14–16. Discusses 2 simple procedures for diagnosing and analyzing children's knowledge of basic facts, level of understanding, and rate of learning facts. Both require materials easily and inexpensively made by a teacher.

Thompson, Charles S., and A. Dean Hendrickson. "Verbal Addition and Subtraction Problems: Some Difficulties and Some Solutions," *Arithmetic Teacher*, XXXIII, No. 7 (March 1986), pp. 21–25. This article alerts the teacher to some of the difficulties that children have in understanding story problems and offers some suggestions.

Thompson, Charles S., and John Van de Walle. "Modeling Subtraction Situations," *Arithmetic Teacher*, XXXII, No. 2 (October 1984), pp. 8–12. Children need to be introduced to different situations that call for different understandings of subtraction.

———. "Paper Dot Plates Give Numbers Meaning," *Arithmetic Teacher*, XXVIII, No. 1 (September 1980), pp. 3–7. Paper plates containing round dots serve to introduce counting, matching, addition and subtraction sentences, and other concepts.

———. "Transition Boards: Moving from Materials to Symbols in Subtraction," *Arithmetic Teacher*, XXVIII, No. 5 (January 1981), pp. 4–7, 9. Transition boards provide a nonproportional model for playing trading games, such as the Banker's Game explained in Chapter 7. The authors show how to use the board to help children learn the decomposition method of subtraction.

Van de Walle, John, and Charles S. Thompson. "A Poster-Board Balance Helps Write Equations," *Arithmetic Teacher*, XXVIII, No. 9 (May 1981), pp. 4–8. A simple poster-board model of a balance scale attached to the chalkboard serves as a means of helping children understand the meaning of *equals*. Either dots or numerals are used in the pans to develop the concept.

Chapter

Teaching Multiplication and Division

In Chapter 10, you will read about

1. three ways to interpret multiplication and models of each way to use with children
2. two types of real-world division situations and ways to model them
3. six strategies for reinforcing basic multiplication and division facts
4. the properties of multiplication and how to help children understand them
5. ways to introduce a multiplication algorithm with manipulative materials
6. using a place-value pocket chart to help children understand multiplication with regrouping
7. using front-end estimation to estimate products
8. using front-end multiplication and aliquot parts to mentally determine exact products
9. the role of zero in multiplication
10. use of multiplication and division patterns to aid in estimating quotients
11. models and a question-answer sequence for developing understanding of a division algorithm

12 five ways that remainders in division can be treated to reflect different situations

One definition of multiplication states that it is the operation on numbers that assigns a single number called the *product* to an ordered pair of numbers called *factors*. Division is defined as the operation used to find a missing factor when one factor and a product are known. Once again we have examples of abstract definitions that are useful at a mature level but that cannot be used with children in the elementary school.

Multiplication is often introduced late in the second grade, although the majority of children's early work with it occurs in the third grade. Division is usually introduced in grade three. Once the two are introduced, work continues through the grades with the most difficult aspects coming in grade five or six. Figure 10-1 shows one sequence for presenting multiplication; while Figure 10-2 shows a sequence for division. Children should also learn to use a calculator for determining answers for multiplication and division.

INTRODUCING MULTIPLICATION

Repeated Addition

Children's early free-play and teacher-directed activities lay foundations upon which an understanding of multiplication can be developed. For example, children who make Unifix-cube trains with alternating cube groups of 2 red, 2 blue, 2 white, 2 red, and so on see a pattern of 2. If children use these cubes to learn to count by twos, they are in fact using a count-on strategy to determine a product. Similar visual patterns are used to develop skip-counting by other numbers, such as 4, 5, and 10.

When the repeated-addition interpretation of multiplication is used to introduce multiplication in grade two or three, the materials and procedures are not new; they form just an extension of an already-known concept to a new situation. By using this concept, you enable children to use what they already know as a foundation upon which to build their understanding of multiplication. For example, when confronted with 4 sets, each of which contains 2 objects, children can first determine the answer by skip-counting by twos 4 times or adding $2 + 2 + 2 + 2$. When you introduce the multiplication expression 4×2, they can see that it is another way to

FIGURE 10-1

A sequence of steps for learning to multiply whole numbers

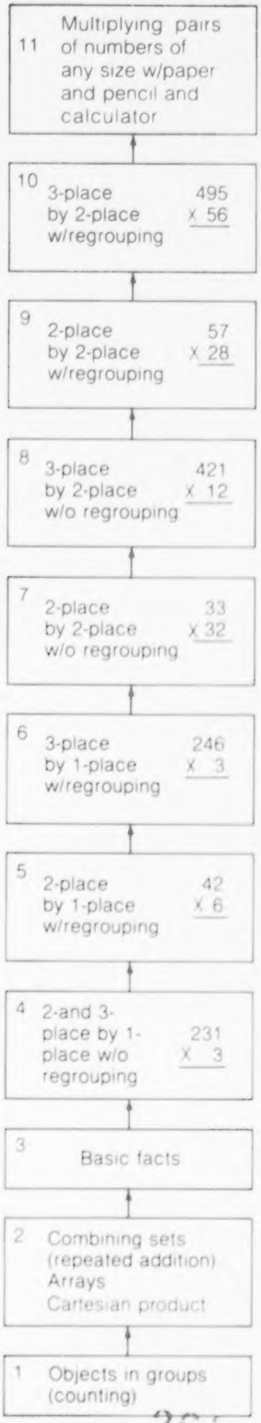
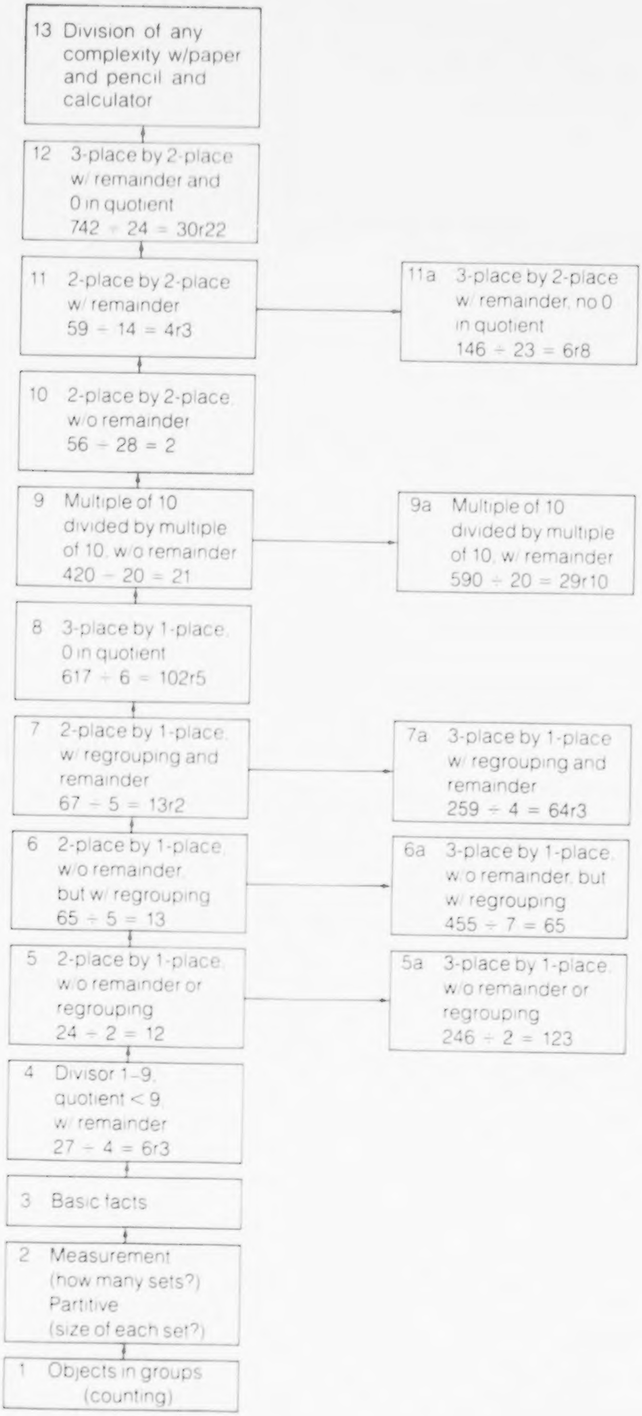

11 Multiplying pairs of numbers of any size w/paper and pencil and calculator
10 3-place by 2-place w/regrouping 495 X 56
9 2-place by 2-place w/regrouping 57 X 28
8 3-place by 2-place w/o regrouping 421 X 12
7 2-place by 2-place w/o regrouping 33 X 32
6 3-place by 1-place w/regrouping 246 X 3
5 2-place by 1-place w/regrouping 42 X 6
4 2-and 3-place by 1-place w/o regrouping 231 X 3
3 Basic facts
2 Combining sets (repeated addition) Arrays Cartesian product
1 Objects in groups (counting)

FIGURE 10-2

A sequence of steps for learning to divide whole numbers


13 Division of any complexity w/paper and pencil and calculator
12 3-place by 2-place w/ remainder and 0 in quotient 742 ÷ 24 = 30r22
11 2-place by 2-place w/ remainder 59 ÷ 14 = 4r3
11a 3-place by 2-place w/ remainder, no 0 in quotient 146 ÷ 23 = 6r8
10 2-place by 2-place, w/o remainder 56 ÷ 28 = 2
9 Multiple of 10 divided by multiple of 10, w/o remainder 420 ÷ 20 = 21
9a Multiple of 10 divided by multiple of 10, w/ remainder 590 ÷ 20 = 29r10
8 3-place by 1-place, 0 in quotient 617 ÷ 6 = 102r5
7 2-place by 1-place, w/ regrouping and remainder 67 ÷ 5 = 13r2
7a 3-place by 1-place w/ regrouping and remainder 259 ÷ 4 = 64r3
6 2-place by 1-place, w/o remainder, but w/ regrouping 65 ÷ 5 = 13
6a 3-place by 1-place, w/o remainder, but w/ regrouping 455 ÷ 7 = 65
5 2-place by 1-place, w/o remainder or regrouping 24 ÷ 2 = 12
5a 3-place by 1-place, w/o remainder or regrouping 246 ÷ 2 = 123
4 Divisor 1–9, quotient < 9, w/ remainder 27 ÷ 4 = 6r3
3 Basic facts
2 Measurement (how many sets?) Partitive (size of each set?)
1 Objects in groups (counting)

express an already familiar situation. Guide them to note that 2 + 2 + 2 + 2 = 8 and 4 × 2 = 8 are both statements about the same situation.

Initially, neither of the terms *repeated addition* and *multiplication* need to be used. Do provide ample opportunities at the concrete-manipulative level for children to understand the new operation. You should plan activities that encourage children's use of familiar materials and devices.

ACTIVITY 10-1

Buying Gum

Begin with this story: "Sally bought gum in packages of 7 sticks each. If she bought 4 packages of gum, how many sticks did she get?"

Allow time for each child to determine the answer, using actual gum packages, coffee-stirrer sticks, or any other materials and procedures. Move among the children to observe their work and encourage promising approaches.

Discuss what different children did. Let those who used sticks of gum, stirrer sticks, or marks drawn on paper explain their work first. If some children used a mathematical procedure, such as repeated addition or perhaps even multiplication, have their explanations follow the first ones.

Give other similar problem situations for the children to solve before you introduce the symbolism for multiplication.

The use of materials to show the repeated-addition concept of multiplication should be spread over a period of days. These materials can be used:

Items from math boxes. Have children organize sets with 4 airplanes in each one. "How many airplanes in 1 set?" "How many in 2 sets?" And so on.

Unifix cubes. Have children join a given number of Unifix cubes. Figure 10-3 shows sets with 3 cubes each. Talk about the number of cubes in 1 set, 2 sets, 3 sets, and so on.

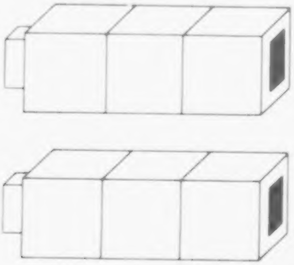


FIGURE 10-3

Unifix cubes linked to show 2 × 3 = 6

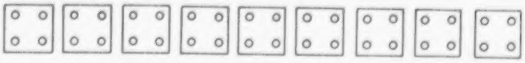


FIGURE 10-4

Set cards used to give children experience with counting by fours. These can be made by sticking adhesive labels to pieces of tagboard or colored railroad board.

Set cards. Have children organize the combinations for a given number as 1 factor. For example, with the cards illustrated in Figure 10-4, children can see that 1 set of 4 is 4, 2 sets of 4 are 8, and so on.

Sets on a magnetic or flannel board. Show equal-sized sets of counters on a magnetic or flannel board, as shown in Figure 10-5. Discuss the fact that this board shows 3 sets of 7 counters each, and determine the total number of objects. Repeat with other sizes of sets.

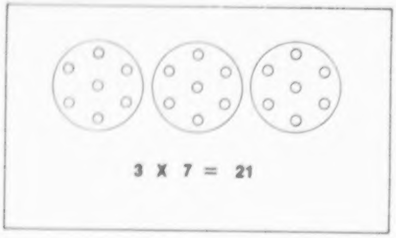


FIGURE 10-5

Markers and symbols used to illustrate the multiplication sentence 3 × 7 = 21

Once you are satisfied that children have the background necessary for understanding the terms and symbolism used for multiplication, you can use one of the activities from an earlier lesson to introduce them. For the gum situation in Activity 10-1, children might have originally determined the answer by counting the gum in four 7-stick packages by using the sentence 7 + 7 + 7 + 7 = 28; or by using the column addition in the margin. Through discussion, lead the children to see common elements in each of the examples: there are 4 packages of gum, each with 7 sticks, and the addend 7 was used 4 times. The multiplication sentence 4 × 7 = 28 applies to this situation. Both 4 and 7 are factors, and 28 is the product. The 4 indicates the number of sets or addends, the 7 the size of each set or addend, and the 28 the number of objects or the sum. Eventually, perhaps in late third grade or fourth grade, the term *multiplier* can be applied to the first factor, *multiplicand* to the second factor, and *product* to the answer.

$$\begin{array}{r} 7 \\ 7 \\ 7 \\ +\ \underline{7} \\ 28 \end{array}$$

Arrays

An array is an arrangement of items in a number of rows, each row containing the same number of items. An array of shoe boxes is shown in Figure 10-6. In it are 4 rows of boxes, each with 12 boxes, or a total of 48 boxes.

FIGURE 10-6

Four rows of shoe boxes with 12 in each row illustrate a 4-by-12 array.

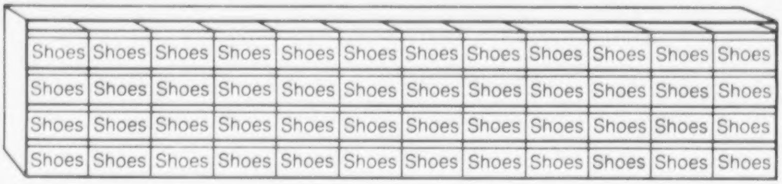

Shoes Shoes Shoes Shoes Shoes Shoes Shoes Shoes Shoes Shoes Shoes Shoes
Shoes Shoes Shoes Shoes Shoes Shoes Shoes Shoes Shoes Shoes Shoes Shoes
Shoes Shoes Shoes Shoes Shoes Shoes Shoes Shoes Shoes Shoes Shoes Shoes
Shoes Shoes Shoes Shoes Shoes Shoes Shoes Shoes Shoes Shoes Shoes Shoes

FIGURE 10-7

A 3-by-5 array

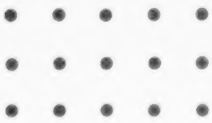


Figure 10-7 shows an array of markers. There are 3 rows, with 5 markers in each row. This is a 3-by-5 array, with the "3" designating the number of rows and the "5" the number of objects in each row. The generally accepted sentence for this array is 3 × 5 = 15, which is read "3 times 5 = 15."

In Chapter 3, learning centers are discussed as a setting for children's work in mathematics. Problem cards are mentioned as one way of directing activities in a learning center. You can introduce the concept of array through a series of problem-card activities. Children work singly or in groups of 2 or 3. Each child has a masonite pegboard with 100 holes arranged in 10 rows of 10 holes each, a box of golf tees, and paper and colored pencils. There should be a set of problem cards for each child or group (Figure 10-8).

FIGURE 10-8

A series of 3 problem cards dealing with arrays

MULTIPLICATION ARRAYS (1)

1. Use 6 red golf tees to make this pattern on your pegboard.

   ○ ○ ○
   ○ ○ ○

   This pattern has 2 rows of tees, with 3 tees in each row. Altogether there are ___ tees.

2. Make this pattern on your pegboard with green tees.

   ○ ○ ○
   ○ ○ ○
   ○ ○ ○

   This pattern has ___ rows of tees, with ___ tees in each row. Altogether there are ___ tees.

3. Make a pattern that has 3 rows, each with 4 yellow tees. Draw the pattern on your paper with a yellow pencil. There are ___ tees in this pattern.

FIGURE 10-8
(continued)

(2)

4. Make a pattern that has 5 rows, with 3 blue tees in each row. Draw the pattern on your paper with a blue pencil. There are ___ tees in this pattern.

5. Each pattern is an array of tees. Another way to write the name of the pattern of blue tees is to call it a 5 by 3 array. The pattern of blue tees is a ___ by ___ array.

6. Use the tees to make these arrays. Draw pictures of arrays on your paper, using any colors you wish.
   a. 2 by 4
   b. 3 by 6
   c. 4 by 5
   d. 5 by 4
   e. 4 by 4
   f. 6 by 2

7. Tell how many tees are in each of the arrays in problem 6.

8. Use your arrays to answer these multiplication sentences:
   a. 2 x 4 = ___
   b. 3 x 6 = ___
   c. 4 x 5 = ___
   d. 5 x 4 = ___
   e. 4 x 4 = ___
   f. 6 x 2 = ___

(3)

9. Use colored pencils to draw some arrays of your own on your paper. Write a multiplication sentence for each of your arrays.

10. Look about the room for patterns that form arrays. Make a picture of the arrays you see, then describe each of your pictures. Write a multiplication sentence for each of your pictures.

Nearly every classroom contains array patterns, such as panes of window glass, rows of ceiling or floor tiles, and sets of lights. If your room has no apparent patterns, you can create some by arranging books on shelves, geometric shapes on a wall board, and pictures on a bulletin board. Once children know about arrays, they should search their room and other parts of the school environment for examples. Children's interest can be heightened by challenging them to find unusual arrays in and around their school.

In addition to the problem-card activities, you can have children work with the following materials:

Adhesive labels. Array patterns made from adhesive labels arranged in rows and columns on colored cardboard provide simple aids for showing arrays. The array for 7 × 7 is shown in Figure 10-9. In (a), 49 labels are arranged in 7 rows with 7 labels in each row. In (b), an array with 7 rows of 1 label each shows a 7-by-1 array. Have children uncover successive columns to see 7 by 2, 7 by 3, and other arrays. In (c), a 1-by-7 array is shown. Uncovering rows now shows 2-by-7, 3-by-7, . . . , 7-by-7 arrays.

Squared paper. One application of the array concept is to find the area of the plane surface enclosed by a rectangular figure. Children develop readiness for the area concept by marking arrays on squared paper. A work sheet with printed directions indicating the arrays to be marked can be used to guide children's activities as they complete arrays like those in Figure 10-10.

FIGURE 10-9

(a) Card showing a 7-×-9 array. (b) The card is covered to show a 7-×-1 array. (c) The card is covered to show a 1-×-7 array.

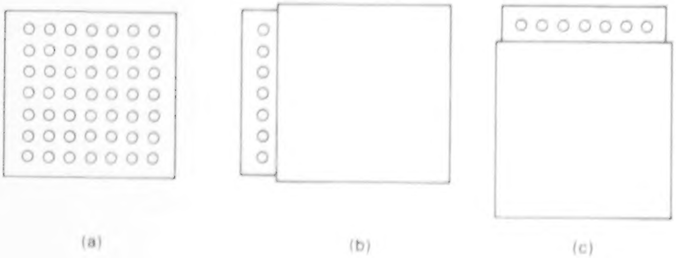
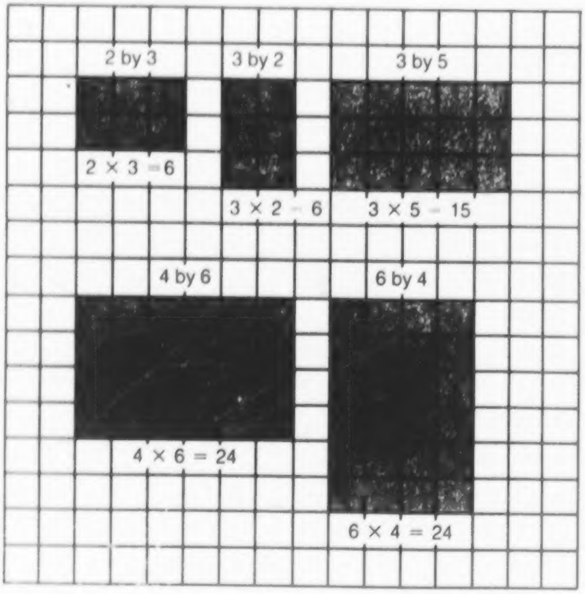


FIGURE 10-10

Samples of arrays drawn on squared paper



Cartesian Products

A Cartesian, or cross, product is determined by matching each element from 1 set with each element from another set to make a set of ordered pairs. Common situations familiar to children can be used to introduce fifth and sixth graders to this concept. Problem 5-13 (Our Favorite Ice Cream Flavors) in Chapter 5 is a Cartesian product situation. To introduce this concept, a situation like this can be used:

ACTIVITY 10-2

Making Bicycles

Tell the children that they are to pretend that they are operating a bicycle manufacturing plant. The plant is capable of producing bicycles with 4 colors: red, silver, blue, and gold. The machines that trim each bicycle can put on white, black, or green trim. How many different combinations of basic colors and trim are there for painting bicycles?

Give pieces of red, blue, silver, and gold paper and white, black, and green paint to pairs of children. Let each pair determine the different combinations that they can make. Have different pairs display one combination each until all combinations have been displayed. Count the number of combinations.

Present other possibilities: 5 basic colors and 3 trim colors, 4 basic colors and 5 trim colors. Have children determine the ordered pairs for each new situation and the total number of combinations.

The Cartesian product interpretation is useful to explain the role of 1 and 0 in multiplication. When one of the sets involved in forming a Cartesian product contains only 1 element, there can be only as many ordered pairs as there are elements in the other set. In the bicycle situation, if there are only gold bicycles, there can be only as many combinations as there are trim colors. It is obvious that if there are no basic-colored bicycles, there can be no combinations of basic color and trim. This fact helps explain why zero times any number is always 0.

There are other situations you can use to develop the Cartesian product concept of multiplication:

> Children can determine how many boy-girl pairs can be matched from a certain set of boys and a certain set of girls. The 2 sets can be called to the front of the room and the ordered pairs of names written on the chalkboard as the children are paired.
>
> Children can determine the number of outfits that can be assembled by matching shirts and pants or skirts and blouses from a person's wardrobe.
>
> Children can determine the number of cake combinations that can be made by matching types of cakes and frostings.

INTRODUCING DIVISION

Two types of situations give rise to division. In one situation, the number of equal-sized subsets in a given set is determined; while in the second, the size of a given number of subsets contained in a larger set is determined. In the first situation—measurement—the size of the original set and the size of 1 subset are known. The number of subsets is to be determined. "How many servings of scrambled eggs can be served from 8 eggs if 2 eggs are used in each serving?" In the second situation—partitive—the size of the original set and the number of subsets are known; the size of 1 subset is to be determined. "How many eggs are in each serving when 8 eggs are shared equally among four people?"

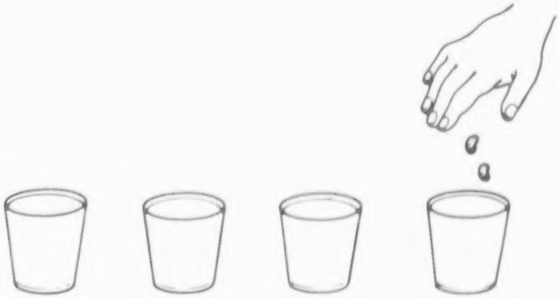
Introducing Measurement Situations

Either of the 2 situations mentioned can be used when division is introduced. Measurement situations are preferred by many mathematics specialists and teachers because they can be related to repeated subtraction. Children see that division modeled by repeatedly removing equal-sized groups is the opposite of multiplication modeled by repeatedly adding equal-sized groups. After studying how primary-grade children learn division, Marilyn Zweng concluded that understanding is enhanced when containers are used so that children actually have places to put the objects when sets are separated.[1] To complete Activity 10-3, third-grade children place beans in paper cups as they investigate measurement situations.

ACTIVITY 10-3

Dividing Beans among Cups

Give each child 15 to 20 lima beans and 9 small paper cups.
Have each child count out 8 beans and hold them in 1 hand.
Direct the children to put the 8 beans into cups, 2 at a time, until all of the beans have been distributed.
"Into how many cups did you put beans?" (4.)



Repeat with other quantities and groupings of beans. Discuss each situation as it is completed to help the children realize that each time they separate a set of beans into groups of known size, they determine the number of cups into which the beans are put.

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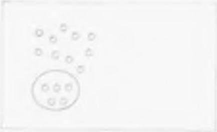
[1] Marilyn J. Zweng, "Division Problems and the Concept of Rate," *Arithmetic Teacher*, XI, No. 8 (December 1964), pp. 547–556.

The following activity is another way to show a measurement situation.

ACTIVITY 10-4

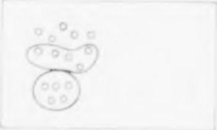
Dividing Marbles among Bags

Use a magnetic or flannel board with circular markers for marbles. "Susie has 15 marbles. She wants to put them into some marble bags. If she puts 5 marbles into each bag, how many bags will she need?" Show 15 disks, and put a ring around 5 of them. Subtract 5 from 15 to show that 10 marbles are left.



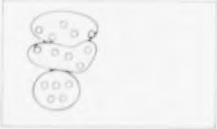
$$\begin{array}{r}15\\-5\\\hline 10\end{array}$$

Put a ring around another 5 disks, and subtract 5 from 10.



$$\begin{array}{r}10\\-5\\\hline 5\end{array}$$

Put a ring around the last 5 disks, and show $5 - 5 = 0$.



$$\begin{array}{r}5\\-5\\\hline 0\end{array}$$

Repeat with other quantities of markers and different groupings.

Introduce the division sentence by discussing already completed activities. The sentence $15 \div 5 = 3$ is used for the marble problem. As the sentence is presented, discuss the meanings of its parts. The first numeral in the sentence indicates the size of the original set of marbles (15); the second numeral indicates the size of each of the equal-sized groups of marbles that were removed from the original set (5); and the last numeral indicates the number of groups of marbles (3). The sentence is read: "Fifteen divided by five equals three."

Partitive Situations

Introduce partitive situations with a familiar situation, such as this.

ACTIVITY 10-5

Partitive Situation

Use a collection of paper dolls and a flannel board divided into 3 sections by horizontal lines representing shelves.

"Juanita has 24 small dolls. She is going to put them in a cabinet that has 3 shelves. If she puts the same number of dolls on each shelf, how many dolls will be on each shelf?"

Discuss with the children ways to determine the answer. One way is to distribute the dolls 1 at a time, putting a doll on each shelf until all are distributed. Children may suggest a faster procedure wherein they place a larger number on each shelf. If dolls are left over, they distribute them 1 at a time, or perhaps 2 at a time if the possibility exists and it occurs to them.

Repeat with other numbers and grouping of dolls.

After children understand partitive situations, help them relate the numbers in a division sentence to such situations. The sentence for the doll problem is $24 \div 3 = 8$. The 24 indicates the size of the original group of dolls; the 3 indicates the number of groups into which the dolls were separated; and the 8 indicates the number of dolls in each group.

Children in grade four and beyond should learn and use the proper terms when referring to numerals in a division sentence. The first number is the dividend, the second number is the divisor, and the answer is the quotient. The meaning of the dividend never changes: It always tells the size of the original set. But the roles of the divisor and quotient are interchanged, depending upon the situation. In a measurement situation, the divisor tells the size of each subset, and the quotient tells the number of subsets. In a partitive situation, the divisor tells the number of subsets, and the quotient tells the size of each subset.

REINFORCING THE LEARNING OF FACTS

Children who are introduced to multiplication and division through appropriate manipulative-material activities and who understand related concepts such as skip counting will learn many of the 100 basic multiplication and 90 division facts as they work. For example, many children see that multiplying by 2 is the same as doubling the factor that is not 2. Multiplication by 5 and 10, and other numbers, is not difficult for children who have had meaningful skip-counting experiences. Once the principles of multiplying by 1 and 0 are understood, they present little difficulty to most children.

When children's work with manipulative materials indicates a good understanding of the repeated-addition or array concept, you should provide activities and materials that will help them learn the basic facts.

Many children develop strategies of their own for remembering multiplication and division facts. These children should be encouraged as long as their strategies lead to accurate results and reasonable speed. You can encourage all children to remember the facts by presenting activities such as the following. A multiplication table is useful for helping fourth- and fifth-grade children realize that there are relatively few difficult combinations to remember.

ACTIVITY 10-6 The Multiplication Table

Begin with a blank multiplication table.

X	0	1	2	3	4	5	6	7	8	9
0										
1										
2										
3										
4										
5										
6										
7										
8										
9										

Stress these points as you and the children complete the table:

- ☐ The factors for each combination are in the top row and the left-hand column.
- ☐ The squares in the first row and the first column of products all contain 0 as the product. "Who can explain why this is so?" (They each have at least 1 zero as a factor.) "How many combinations with a 0 product are there?" (19.)
- ☐ The second row and second column of products contain numbers that are the same as the factor that is not 1 (except for 1 × 1). "Who can explain why that it so?" (Because 1 times any number is that number. One is the identity for multiplication.) "How many of these combinations are there?" (17.) Remind the children that if they remember the rules for multiplying by 1 and by 0, they know more than one-third of all the multiplication facts. (36.)

- Combinations in the next row and column have 2 as 1 factor. The product is twice the number that is not 2. There are 15 of these combinations. Add these to the 36 already discussed and children will see that they know more than half of the 100 facts.

X	0	1	2	3	4	5	6	7	8	9
0	0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8	9
2	0	2	4	6	8	10	12	14	16	18
3	0	3	6	9	12	15	18	21	24	27
4	0	4	8	12	16	20	24	28	32	36
5	0	5	10	15	20	25	30	35	40	45
6	0	6	12	18	24	30	36	42	48	54
7	0	7	14	21	28	35	42	49	56	63
8	0	8	16	24	32	40	48	56	64	72
9	0	9	18	27	36	45	54	63	72	81

- Point out that there are 13 combinations that have 5 as at least 1 factor. Most children learn these easily. These with the others account for nearly two-thirds of all the facts.
- Determine if there are other combinations that children know. If there are, determine what and how many there are; add the number to 64.
- Before you conclude the activity, identify the relatively few facts that can be considered the hard facts. (The list will vary among class members.)

Combinations that involve 9 as a factor are difficult for many children to learn. The following 2 teacher-led activities will help children develop strategies for dealing with 9.

ACTIVITY 10-7

Nines Are Easy

Begin by discussing these questions:

- "6 tens are?" (60.) "Six nines will be less; the answer is in the fifties."
- "4 tens are?" (40.) "Four nines will be less; the answer is in the thirties."

  Repeat with other tens and nines until the children recognize the pattern.

Write these combinations on the board:

$9 \times 6 = 54$ $7 \times 9 = 63$
$9 \times 3 = 27$ $8 \times 9 = 72$
$9 \times 4 = 36$ $5 \times 9 = 45$

"Look at the number in the tens place of each product. How does it compare with the number being multiplied by 9?" (It is 1 less.)
Show these combinations:

9×4 6×9
7×9 9×8
9×9 2×9

"What is the number in the tens place for each of these combinations?" (It is 1 less than the number being multiplied by the 9.)[2]

ACTIVITY 10-8 Products with Nine as One Factor

Show cards containing numerals such as 4, 3, and 7. "What do we add to each of these numbers to make 9?" (5, 6, and 2.)
Write some facts with 9 on the chalkboard:

$9 \times 4 = 36$ $8 \times 9 = 72$
$9 \times 7 = 63$ $6 \times 9 = 54$
$9 \times 5 = 45$ $2 \times 9 = 18$

"If we add the digits in each product, what is the sum?" (9.) Tell the children that in the nines facts, the sum of the digits is always 9.
Show some partially completed facts:

$9 \times 3 = 2\_\_$ $7 \times 9 = 6\_\_$
$9 \times 5 = 4\_\_$ $8 \times 9 = 7\_\_$

Point out that the tens digit has been written. "Because the sum of the digits in the product must equal 9, what must be in the ones place of each product?"
Repeat with all combinations of facts with 9.[3]

Help children to combine the processes in these 2 activities to determine the product when 9 is a factor. The number in the tens place is 1 less than the factor being multiplied by 9; the number in the ones place is the number that is added to the tens place number to give a sum of 9. Follow discussions with work sheets to give practice on the facts just considered. Work sheets containing examples like those in Activities 10-7 and 10-8 give practice with multiplying by 9.

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[2] This activity is adapted from Larry Leutzinger, *Strategies for Learning the Basic Facts* (Cedar Falls: Iowa Council of Teachers of Mathematics, 1981), p. 48.

[3] Adapted from Larry Leutzinger, p. 50.

The process of factoring can serve as an aid in learning the multiplication facts, even for children who have not been formerly introduced to the term or to factoring processes. For example, children who are familiar with arrays and are shown this 4-by-6 array

will readily recognize that it can be separated into two 2-by-6 arrays. This knowledge can be used to determine the product of 4 times 6: 2 times 6 is 12, 2 twelves are 24. By showing children other arrays involving 4, 6, and 8 and having them separate the arrays into easily handled parts, you can encourage them to think of some of the more difficult facts in terms of doubling. With such encouragement, a child learns to think: $8 \times 8 = \square$ is the same as $4 \times 8 = \square$ twice; 4 times 8 is 32, so 8 times 8 is twice 4 times 8; when I double 32, I get 64, so 8 times 8 is 64.

Strategies for Divison

Capitalize on your children's knowledge of multiplication to help them learn division facts. Consider division to be a process of finding a missing factor when a product and 1 factor are known. Focus on a product and its factors at first:

ACTIVITY 10-9

Missing Factors

"What numbers do we multiply to get 24?"
Consider both $4 \times 6 = 24$ and $3 \times 8 = 24$. ($2 \times 12 = 24$ is not a basic fact.)
"What numbers do we multiply to get 36?"
Consider both $4 \times 9 = 36$ and $6 \times 6 = 36$. ($3 \times 12 = 36$ and $2 \times 18 = 36$ are not basic facts.)
Repeat with other numbers that have 2 factors less than 10.

Next focus on a product and a known factor:

ACTIVITY 10-10

One Missing Factor

"What is the other factor when we have 56 and 7?" (8.)
"What is the other factor when we have 48 and 8?" (6.)
Repeat with other products and known factors.

Use work sheets to give practice with division facts. First show a missing factor example; then use the related division sentence:

$$36 = 4 \times \square \qquad 36 \div 4 = \square$$
$$49 = 7 \times \square \qquad 49 \div 7 = \square$$
$$63 = 7 \times \square \qquad 63 \div 7 = \square$$

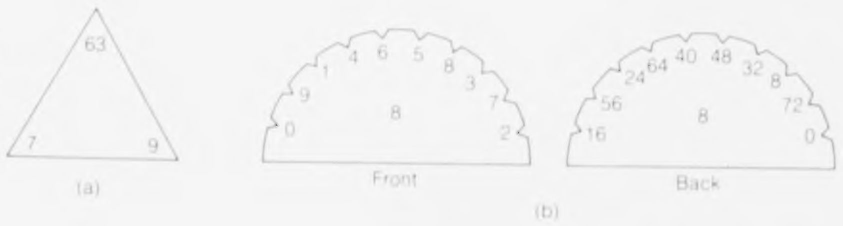
The multiplication table in Activity 10-6 can also be used for learning division facts. To determine the quotient for $56 \div 7 = \square$, locate "56" in the column beneath "7" in the top row of factors; then move across the table to the left column, where the quotient, 8, is located to the left of "56." To find the quotient for $56 \div 8 = \square$, locate "56" in the column beneath "8" in the top row; then move across to "7" in the left column. This use of the table is consistent with the way it is used to learn the multiplication facts.

Practice Materials

Flash cards that emphasize the relationship between multiplication and division can be used to provide practice with the basic facts. The card in Figure 10-11(a) is used this way: For multiplication, cover the "63" and show the 2 factors; the child names the product. For division, cover 1 of the factors and show the other factor and product; the child names the missing factor. The card in Figure 10-11 has all the division and multiplication facts in which 8 is 1 factor. It is used by children working in pairs, one being the "teacher" and

FIGURE 10-11

Two types of flash cards for practicing multiplication and division facts



other the "student." The "teacher" puts a pencil in a notch to indicate the number combination to use. When the front is facing the "student" and the pencil is put in the "6" notch, the combination is 6 times 8, to which the "student" responds by giving the product, 48. The answer is on the card's back. For division, the card is turned around so that the "student" sees a product and the "8." Again the "teacher" puts a pencil in an notch to indicate which pair of numbers to consider. The "student" names a factor when shown a product and "8." The procedures for using practice materials for addition and subtraction apply here. That is, if a child cannot readily name a product or quotient, say it or show it immediately. Then show the combination and ask the child to name the correct product or quotient.

A well-equipped classroom will contain other practice materials as well. Children's textbooks contain pages that feature basic-fact practice. These can be used after children become familiar with both operations and have learned strategies for dealing with them. Computer programs designed to provide instruction and practice with operations typically include sections dealing with the basic facts for multiplication and division. For example, *RoboMath*[4] has multiplication and division practice in the *RoboMath* arcade. Preprogrammed devices, such as Educational Insights' "Charlie"[5] have preprogrammed cards for multiplication and division.

A calculator makes it easy to show relationships between multiplication and addition, division and subtraction, and multiplication and division. To show multiplication as repeated addition, use 1 factor as an addend and add it as many times as indicated by the other factor. For $7 \times 9 = \square$, use 9 as the addend 7 times to get the product 63. Compute by multiplying on the machine to verify the answer.

In a similar way, show division as repeated subtraction. Set the dividend, such as 64, into the machine. Subtract 8 as many times as needed to reach 0 or a number smaller than the divisor. Count each time the $\boxed{=}$ key is pressed to get the quotient. When children alternate these 2 operations with the same set of numbers, they have achieved another way of seeing that multiplication and division are inverse operations. At the time they are doing this, children get valuable practice with basic facts.

TEACHING SOME PROPERTIES OF MULTIPLICATION AND DIVISION

The Commutative Property

The commutative property of multiplication is readily observed by children when they use arrays, because many opportunities are available for them to

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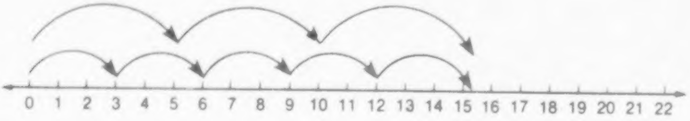
[4] *Robomath* is a product of Mindplay, 82 Montvale Avenue, Stoneham, MA 02180.

[5] Educational Insights, 19560 S. Rancho Way, Dominguez Hills, CA 90220

FIGURE 10-12

Number line used to show that 5 × 3 = 3 × 5



see that changing the order of a pair of factors does not affect their product. For example, the 2-by-3 and 3-by-2 arrays on squared paper in Figure 10-10 show that 2 × 3 = 6 and 3 × 2 = 6. You can include exercises on problem cards to call children's attention to the commutative property as they work with the pegboards and golf tees and the squared paper and colored pens and pencils.

The number line is a useful aid for understanding the commutative property. When an open sentence with a pair of small factors is displayed, a child can put arrows above the number line to illustrate the sentence and show the product (Figure 10-12). The factors in the sentence should then be reversed and the meaning of the new sentence demonstrated.

Division is not a commutative operation. Children who understand that multiplication is commutative can see that division is not commutative by examining a few examples such as 72 ÷ 9 = ☐ and 9 ÷ 72 = ☐. These examples make it clear that changing the order of the numbers changes the answer.

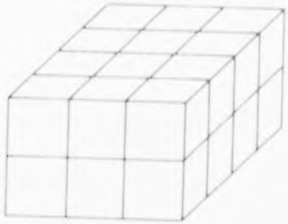
The Associative Property

An introduction to the associative property of multiplication should help children recognize its significance. This property frees them to deal with pairs of factors in any way they choose in operations involving 3 or more factors.

The associative property is of little significance until children begin work with 3 factors. They will do this when they determine volumes of solid figures, such as rectangular prisms, where numbers indicating length, width, and height are multiplied. The volume of a rectangular prism with a length of 4, a width of 3, and a height of 2 (Figure 10-13) can be determined in several ways:

FIGURE 10-13

A 2 × 3 × 4 rectangular solid



$$\begin{aligned} 3 \times 4 \times 2 &= 24 \\ 2 \times 4 \times 3 &= 24 \\ 4 \times 2 \times 3 &= 24 \\ 3 \times 2 \times 4 &= 24 \end{aligned}$$

(There are still other combinations that you should identify before continuing your reading.) Regardless of which pair of factors is multiplied first, the product is always 24. Work with this type of multiplication is discussed further in Chapter 13 in connection with finding the volume of three-dimensional figures. (You should recognize that when the order of the factors is changed, the commutative property is used along with the associative property.)

Another way to show children the usefulness of the associative property is to present combinations such as these:

$$\begin{aligned} 4 \times 9 \times 25 &= \square \\ 36 \times 2 \times 50 &= \square \end{aligned}$$

In the first example, the product is easily determined when the product of 4×25 is named first; the 100 can then be multiplied by 9. In the second, determine the answer for 2×50, and then multiply 100 by 36.

The Distributive Property

The distributive property of multiplication over addition can be taught before all the multiplication facts have been introduced. This knowledge provides a useful strategy for studying multiplication facts with larger products. It can be used to find the answer to $4 \times 8 = \square$ if the multiplication fact $4 \times 4 = 16$ is known. The 8 in $4 \times 8 = \square$ can be renamed as $4 + 4$ and the sentence expressed as $4 \times (4 + 4) = \square$. The answer is then determined by completing the sentence

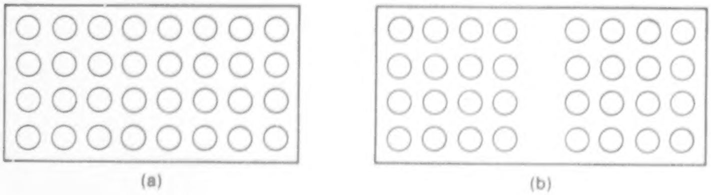
$$4 \times (4 + 4) = \square \rightarrow (4 \times 4) + (4 \times 4) = \square \rightarrow 16 + 16 = 32.$$

Before children can be expected to understand how to apply the property in this way, they should use aids that will make the property meaningful to them. For example, an array can be arranged on a magnetic board for the sentence $4 \times 8 = \square$ (Figure 10-14). Children first verify that it is a 4-by-8 array (a). Then rearrange the array as in (b), and display it to show

$$(4 \times 4) + (4 \times 4) = \square$$

The children determine the products for the 2 expressions of 4×4, and the sum of the products is determined. The sequence of steps in the sentence is

$$4 \times 8 = 4 \times (4 + 4) = (4 \times 4) + (4 \times 4) = 16 + 16 = 32.$$



(a) (b)

FIGURE 10-14

Arrays used to illustrate the distributive property. (a) 4 × 8 = 32. (b) (4 × 4) + (4 × 4) = 16 + 16 = 32.

because the second factor, 8, has been renamed 4 + 4. This sentence now corresponds to the array shown in (b). Finally they should see the original array (a) again and its sentence:

$$4 \times 8 = \square \rightarrow 4 \times 8 = 32.$$

By seeing different arrangements of the 4-by-8 array, children will recognize that 8 can be renamed in other ways, such as 6 + 2 and 5 + 3, and that the sentence can be expressed as

$$4 \times (6 + 2) = (4 \times 6) + (4 \times 2) = 24 + 8 = 32$$

and

$$4 \times (5 + 3) = (4 \times 5) + (4 \times 3) = 20 + 12 = 32.$$

The Identity Element

An identity element is a number that, when operated on with another number, results in an answer that is the same as the second number. *One* is the identity element for multiplication.

The role of 1 is also important in division. You should provide children with activities that help them generalize that whenever a number is divided by 1, the quotient is always the number that is divided. They should also learn that any number divided by itself results in the quotient of 1, with 0 divided by 0 excepted.

The Role of Zero in Multiplication and Division

Zero has a special role in multiplication and division. In multiplication, the product is always 0 when 0 is 1 of the factors. The role of 0 in division is a special one. When the dividend is 0, the quotient is also 0 if the divisor is a number other than 0. This is illustrated by the example 0 ÷ 9 = 0. This sentence is true, because 0 × 9 = 0. However, 0 is never used as a divisor.

INTRODUCING A MULTIPLICATION ALGORITHM

Historically there have been several algorithms for multiplying pairs of numbers greater than 9. The algorithm shown here is used most often:

$$\begin{array}{r} 32 \\ \times\ 2 \\ \hline 64 \end{array}$$

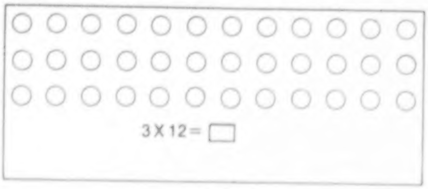
Introduce this algorithm by giving children a problem situation to solve.

ACTIVITY 10-11

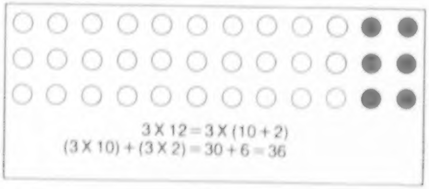
Introducing an Algorithm

Tell this story: "Billy was arranging some books in a new bookcase. He found that 12 books fit on each of the 3 shelves. How many books did the bookcase hold?"

Talk about models that can be used to represent the books in this situation. Markers on a magnetic board, a place-value pocket chart, and Cuisenaire rods are shown here.



(a)

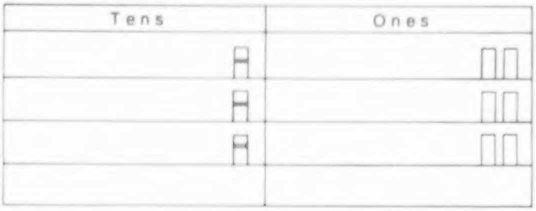


(b)

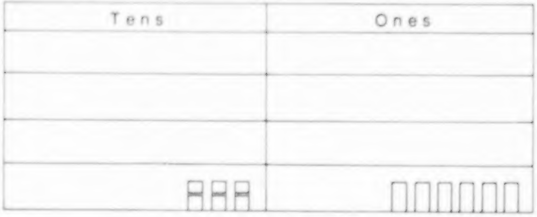
Discuss the multiplication sentence

$$3 \times 12 = 36$$

There is a 3-by-12 array on the magnetic board; the pocket chart shows 3 pockets, each with a bundle of 10 and 2 single cards; there are 3 sets of

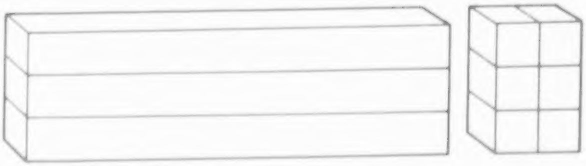


(a)



(b)

(a)



(b)

Cuisenaire rods, each with 1 tens rod and 2 ones rods. Relate the sentence to each concrete model, making certain that the children see the tens and ones in each model.

Next show the algorithm form:

$$\begin{array}{r} 12 \\ \times\ 3 \\ \hline \end{array}$$

Show that there is a total of 6 ones in each model, and relate this fact to multiplication of 2 by 3 in the algorithm. Record the answer:

$$\begin{array}{r} 12 \\ \times\ 3 \\ \hline 6 \end{array}$$

Show that there are 3 tens, or 30, in each model, and relate this to the multiplication of 10 by 3. Record the answer:

$$\begin{array}{r} 12 \\ \times\ 3 \\ \hline 6 \\ 30 \end{array}$$

Finally talk about the total in each model. Complete the algorithm:

$$\begin{array}{r} 12 \\ \times\ 3 \\ \hline 6 \\ 30 \\ \hline 36 \end{array}$$

Repeat with other similar problem situations.

Another way to show the algorithm employs expanded notation and may make the work clearer for some children:

$$\begin{array}{l} \quad 10 + 2 \\ \underline{\times \qquad\ 3} \\ 30 + 6 + 36 \end{array}$$

As children work, they see arrays, structured materials, and pocket charts illustrating the meaning of the algorithm. It is recommended that different models be used, because a model that is meaningful to 1 child may not have the same meaning for another. Under your careful guidance, most children will have the meaning of each step of the algorithm clarified by the models.

After multiplication with numbers like the ones used above, examples that have numbers larger than 100 as multiplicands and require no regrouping can be introduced. No particular difficulties are involved in this multiplication. A few examples like these

$$\begin{array}{r} 123 \\ \times\ 3 \\ \hline \end{array} \qquad \begin{array}{r} 241 \\ \times\ 2 \\ \hline \end{array} \qquad \begin{array}{r} 342 \\ \times\ 2 \\ \hline \end{array}$$

can be demonstrated with a pocket chart or structured materials.

Writing numerals in the algorithm in the expanded form should be discontinued once the use of the algorithm for multiplication is understood. Children will realize that the short, or common, form of the algorithm is an efficient way to do multiplication.

MULTIPLICATION WITH REGROUPING

Children who understand the meaning of expanded notation and regrouping with addition and subtraction and who also understand simpler multiplication should have no particular difficulties learning to use the algorithm for multiplication with regrouping.

As in all cases when new steps in a process are presented, you should employ meaningful materials to introduce multiplication with regrouping. Children should use these materials as they explore and discuss the meanings of the steps involved in regrouping. The use of the place-value pocket chart and array will be described.

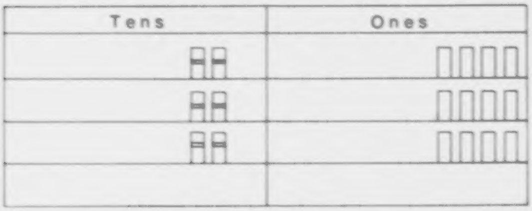
ACTIVITY 10-12 Regrouping in Multiplication

Put a sentence and algorithm involving 3 and 24 on the chalkboard:

$$3 \times 24 = \square \qquad \begin{array}{r} 24 \\ \times\ 3 \\ \hline \end{array}$$

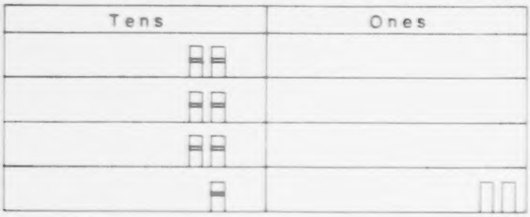
Ask each child to make up a problem situation for this multiplication. Example: "There are 3 boxes of candy bars; each as 24 bars. How many candy bars are in the 3 boxes?"

Have a child represent the situation with a place-value pocket chart.



$$\begin{array}{r} 20 + 4 \\ \times \qquad 3 \\ \hline 12 \end{array}$$

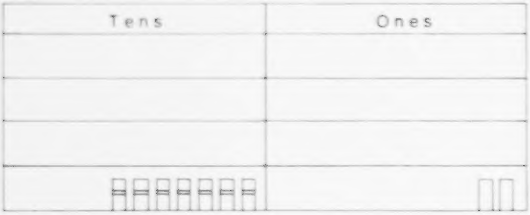
As a child combines the markers in the ones place of the chart, have another child do the multiplication in the ones place of an expanded algorithm.



$$\begin{array}{rr} 20\ + & 4 \\ \times & 3 \\ \hline 60\ + & 12 \end{array}$$

$$\begin{aligned} &= 60 + (10 + 2) \\ &= (60 + 10) + 2 \\ &= 70 + 2 \\ &= 72 \end{aligned}$$

Have a child combine the bundles in the tens place while another multiplies 3 times 20 in the algorithm.
Rename the 12 as 10 + 2, then add the 10 and 60; complete the addition of the partial products.



Repeat with other pairs of factors.

Arrays are also useful for illustrating multiplication with regrouping. To show the meaning of 7 × 14 = ☐, prepare 2 arrays, 7-by-10 and 7-by-4. Place the arrays together to show a 7-by-14 array, as in Figure 10-15(a). Then the array can be separated to show the 7-by-10 and 7-by-4 arrays (b). The sentence and algorithm should be rewritten as

$$7 \times (10 + 4) = \square \qquad \text{and} \qquad \begin{array}{r} 10 + 4 \\ \times 7 \\ \hline \end{array}$$

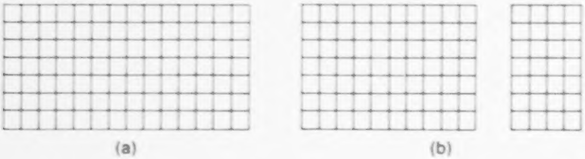


FIGURE 10-15

Array used to show the distributive property with the sentence 7 × 14 = ☐. (a) Array shows the original sentence. (b) Array is separated into two parts to show 7 × (10 + 4) = (7 × 10) + (7 × 4) = 70 + 28 = 98.

The steps to show multiplication with the sentence and algorithm are the same as the ones used in Activity 10-12. Other arrays should be prepared and used in the same manner.

Naturally children should eventually learn the standard algorithm for completing this type of multiplication. Instruction might follow this progression:

$$\begin{array}{rl} 37 \rightarrow & (30 + 7) \\ \underline{\times 2} \rightarrow & \underline{\qquad +2} \\ & 60 + 14 \\ = & 60 + (10 + 4) \\ & (60 + 10) + 4 \\ = & 70 + 4 \\ = & 74 \\ & \text{(a)} \end{array} \qquad \begin{array}{r} 37 \\ \underline{\times 2} \\ 14 \\ \underline{60} \\ 74 \\ \\ \\ \text{(b)} \end{array} \qquad \begin{array}{r} 37 \\ \underline{\times 2} \\ 74 \\ \\ \\ \\ \\ \text{(c)} \end{array}$$

In (a) the multiplication is completed using the expanded form; in (b) it is completed with both partial products written in a vertical placement; and in (c) it is completed with the standard algorithm.

MULTIPLICATION WITH TWO NUMBERS LARGER THAN TEN

Sequence of Steps

$$\begin{array}{r} 53 \\ \underline{\times 36} \\ 318 \\ \underline{159} \\ 1908 \end{array}$$

The commonly used algorithm for multiplying 2 numbers larger than 10 is shown in the margin. Build on children's knowledge of expanded notation to help them understand this algorithm. When 12 and 14 are factors, the algorithm can be written as

$$\begin{array}{r} 14 \\ \underline{\times 12} \end{array}$$

$$\begin{array}{r} 14 \\ \underline{\times 12} \\ 8 \\ 20 \\ 40 \\ \underline{100} \\ 168 \end{array}$$

To multiply these numbers, children should think of the multiplier as 10 + 2. When the multiplication has been completed, there will be 4 partial products, as shown in the margin. By seeing each of the partial products separately, children are more likely to recognize that there are 4 separate pairs of numbers to multiply. Later the process is shortened so that only 2 partial products are written.

When partial products are indicated in the algorithm in the most sophisticated form, the "0" is not written in the ones place of the second partial product. Children must realize that it is omitted only as a time-saving technique (see previous example in margin). The "14" is indented in the second partial product to show that it represents 14 tens, so it is placed with the "4" in the tens place.

$$\begin{array}{r} 14 \\ \times 12 \\ \hline 48 \\ 140 \\ \hline 168 \end{array}$$

Regrouping

$$\begin{array}{r} 14 \\ \times 12 \\ \hline 28 \\ 14 \\ \hline 168 \end{array}$$

$$\text{(a)}\quad \begin{array}{r} 36 \\ \times 48 \\ \hline 48 \\ 240 \\ 240 \\ 1200 \\ \hline \end{array}$$

$$\text{(b)}\quad \begin{array}{r} {}^{24}36 \\ \times 48 \\ \hline 288 \\ 144 \\ \hline 1728 \end{array}$$

Multiplication involving numbers greater than 10 that require regrouping is taught after children understand how to multiply 2 such numbers with no regrouping. As the first few examples of multiplication with regrouping are encountered, children should complete them by writing each partial product in the algorithm, as in the margin (a). Then the combined partial products should be written; the 4 tens from the 48 are added to the 24 tens, giving the first partial product of 288. The regrouping of the 2 hundreds in the second partial product should be noted too. Memory aids, as shown in the example in the margin (b), might be used in the algorithm to help children remember the numbers that are being regrouped. After it has been used, the "4" is marked out to prevent confusion when the "2" is written later. You should also assign frequent adding-by-endings exercises that involve the products of 2 numbers smaller than 10 and the numbers 1 through 8. These experiences should include the use of work sheets and oral exercises, as indicated in Chapter 9, to give children practice with the type of addition required in multiplication with regrouping.

Once children understand the process of multiplying 2 numbers that require regrouping, they can move on to work with larger numbers. The steps in the process are the same as those with numbers expressed by 2-digit numerals. You should continually observe children as they work examples involving larger numbers to prevent errors and the formation of incorrect habits.

ESTIMATION AND MENTAL COMPUTATION

It was pointed out during the discussion of estimating sums in Chapter 9 that the process of front-end estimation is a useful strategy. The same strategy can be applied to making estimates for multiplication. The usual procedure for determining the product for this multiplication

$$\begin{array}{r} 348 \\ \times \quad 4 \\ \hline \end{array}$$

is to multiply 4 times the number in the ones place, the tens place, and the hundreds place in order. A front-end estimate begins in the largest place-value position rather than the smallest. For 4 × 348, think "4 times 300 is 1200" to estimate the answer. In this example, front-end estimation using just the largest place-value position gives the same result as rounding the number 348 to the nearer hundred and multiplying the result by 4. The front-end estimate can be improved by adding to it the product of 4 times 40, giving an estimate of 1360. This estimate is slightly less than the actual product. If rounding off is used to estimate the product, 348 is rounded to the nearer 10 and multiplied by 4, giving the estimated product 1400, which is slightly more than the actual product. Children should be introduced to both estimation processes.[6]

The front-end process can also be used to compute exact answers mentally in multiplication. To multiply 4 times 34 mentally, the thinking is: 4 times 30 equals 120, 4 times 4 equals 16, 120 + 16 = 136. Encourage children to multiply mentally by showing them how to use front-end multiplication with a set of numbers like these:

$$\begin{array}{r} 26 \\ \times\ 7 \\ \hline \end{array} \quad \begin{array}{r} 42 \\ \times\ 8 \\ \hline \end{array} \quad \begin{array}{r} 94 \\ \times\ 6 \\ \hline \end{array} \quad \begin{array}{r} 87 \\ \times\ 5 \\ \hline \end{array} \quad \begin{array}{r} 62 \\ \times\ 7 \\ \hline \end{array}$$

A process dealing with aliquot parts of numbers, an old method applicable only to certain numbers, is another mental multiplication process that children can learn. Children's knowledge of money can be used as a means of introducing this mental process.

ACTIVITY 10-13

Mental Multiplication with Aliquot Parts

Present the multiplication $24 \times 5 = \square$. Think of 24 nickels. Exchange 2 nickels for a dime.

$$\begin{array}{cccc} 5¢ & 5¢ & \rightarrow & 10¢ \\ 5¢ & 5¢ & \rightarrow & 10¢ \\ 5¢ & 5¢ & \rightarrow & 10¢ \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ 5¢ & 5¢ & \rightarrow & 10¢ \end{array}$$

Think: 24 nickels equals (24 ÷ 2) dimes, so 24 × 5 equals (24 ÷ 2) times 10, or 12 times 10, which is 120. Multiply 5 by other even numbers for additional practice.

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[6] For further information about front-end estimation, see Barbara S. Reys, "Teaching Computational Estimation: Concepts and Strategies," in *Estimation and Mental Computation* (Reston, Va.: National Council of Teachers of Mathematics, 1986), p. 38.

Present the multiplication 24 × 25 = ☐. Think 24 quarters.

Exchange 4 quarters for one dollar.

Think: 24 quarters equals (24 ÷ 4) dollars, so 24 × 25 equals (24 ÷ 4) times 100, or 6 times 100, which is 600. Multiply 25 by other even numbers for additional practice.

Encourage children to think of other numbers to which this process will apply. (To multiply by 50, divide by 2, then multiply by 100; for 250, divide by 4 and multiply by 1000.)[7]

ZERO IN MULTIPLICATION

Factors such as 306 and 4,002, especially when they appear in the multiplier, present difficulties to some children. Before multiplying numbers containing 0, multiplication by 10 and its multiples should be mastered. Sets, arrays, the number line, and other aids should be used until children can generalize that when 10 is the multiplier, the product is the multiplicand with a 0 annexed. (An alternate generalization is to think of the process as moving the number 1 place to the left and placing a 0 in the ones place.) To multiply by a number such as 20 or 30 as in the sentence 20 × 6 = ☐, children should be able to think "2 × 6 = 12, annex 0 to give 120." Computation with other powers of 10 and their multiples should be taught in a similar manner so that children can generalize about them too. After they have learned this process, frequent oral exercises reinforce children's ability to use it. Efficient use of this process is important in multiplication and essential before beginning to learn the division algorithm.

In multiplying the first problem in the margin, children should be able to determine the second partial product by multiplying by the 2 hundreds (or 20 tens) in 1 step rather than 2. There is no reason for children to write the work with 3 partial products as shown in the second algorithm in the margin.

$$\begin{array}{r} 346 \\ \times 209 \\ \hline 3114 \\ 6920 \\ \hline 72314 \end{array}$$

$$\begin{array}{r} 346 \\ \times 209 \\ \hline 3114 \\ 000 \\ 692 \\ \hline 72314 \end{array}$$

CALCULATOR CHALLENGES

Children who understand the conventional algorithm for multiplying with whole numbers can be challenged to use a calculator to complete activities such as the two that follow.

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[7] This activity is adapted from Jack A. Hope, "Mental Multiplication: Aliquot Parts," *Arithmetic Teacher*, XXXIV, No. 3 (November 1986), pp. 16–17.

Multiplying 27

3 × 27
6 × 27
9 × 27
12 × 27
15 × 27
18 × 27
21 × 27
24 × 27
27 × 27
30 × 27

Complete the products for the examples in the margin. Note the sequence of numbers in the hundreds and ones places of the products and the sequence of numbers in the tens places. Select any product, say, 486 (18 × 27). Rearrange the numerals by putting the "6" before the "48." Is 648 divisible by 27? Put the "4" after "86." Is 864 divisible by 27? Rearrange other products the same ways. Are all of the rearranged numbers divisible by 27?

Russian Peasant Multiplication

This old multiplication process is made simple by the calculator. The process is illustrated with two examples. In the left-hand column of each example the factor is divided by 2, and the quotient is recorded. (If the calculator shows a decimal fraction quotient, only the whole number portion is recorded.) Divide each successive quotient by 2 until 1 is reached. The factor in the right-hand column of each example is multiplied by 2 and its product recorded beneath it. Each successive product is multiplied by 2. Certain rows are marked out. What criterion is used to determine the rows to mark out? What is done with the remaining numbers in the right-hand column to determine the product?

~~48~~	~~×~~	~~28~~	37	×	57
~~24~~		~~56~~	~~18~~		~~114~~
~~12~~		~~112~~	9		228
~~6~~		~~224~~	~~4~~		~~456~~
3		448	~~2~~		~~912~~
1		896	1		1824
		1344			2109

INTRODUCING A DIVISION ALGORITHM

$$\begin{array}{r} 46 \\ 3\overline{)138} \\ \underline{12} \\ 18 \\ \underline{18} \end{array}$$

The standard division algorithm for the sentence 138 ÷ 3 = 46 is shown in the margin. Before children can perform it with full understanding, they must know and understand certain basic concepts and skills. These include: (1) knowledge of both partitive and measurement division situations, (2) knowledge of the multiplication and division facts, (3) ability to subtract, and (4) ability to multiply and divide by 10 and its powers and multiples.

Dividing by Numbers Smaller Than Ten

Situations that can be used to promote understanding of the division algorithm abound. A problem such as the following is useful to introduce division by a number smaller than 10:

ACTIVITY 10-14

A Division Algorithm

- "Twenty-eight marbles will be shared equally by 6 children. How many marbles will each one get?"
- Use 28 real marbles or disks on a magnetic or flannel board. Then form 6 equal-sized groups. There will be 4 objects in each group, with 4 left over.
- Show an algorithm for this situation:

$$\begin{array}{r} 4 \\ 6\overline{)28} \\ \underline{24} \\ 4 \end{array}$$

- Help children develop a strategy for determining quotients without concrete aids. One strategy is to relate division to multiplication. For this example, use these facts:

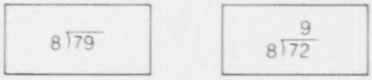
$$6 \times 1 = 6 \quad 6 \times 2 = 12 \quad 6 \times 3 = 18 \quad 6 \times 4 = 24 \quad 6 \times 5 = 30$$

- Questions such as "Will each child get at least 1 marble?" "At least 2 marbles?" "At least 3 marbles?" "At least 4 marbles?" "At least 5 marbles?" help children see that *4 marbles* is the answer. (The 4 marbles that are left over cannot be divided among 6 children.)
- Repeat with other situations.

Think-back flash cards are helpful for developing children's skills in determining quotients. The card for $8\overline{)79}$ is shown in Figure 10-16. The front of the card (a) shows the division algorithm without the quotient, and the back (b) shows the basic fact associated with the division. A child is shown $8\overline{)79}$ and then thinks back to the related division fact and says,

FIGURE 10-16

Example of a think-back flashcard. (a) Front shows the division algorithm. (b) Back shows the associated basic fact.



(a) (b)

"Seventy-two divided by 8 equals 9." (Avoid the expression "8 goes into 72 nine times.") If a child cannot think back to the correct basic fact, reverse the card so it can be read. Then show the front of the card again and have the child repeat the statement of the basic fact. Group and individual work with think-back cards should be provided frequently until children are skillful in naming quotients.

Multiplication and Division Patterns

Skill in multiplying and dividing by 10 and its powers and multiples is needed in order to estimate quotients or use divisors greater than 10. Patterns such as the following can be used to develop this skill:

$1 \times 1 = 1$	$1 \times 10 = 10$	$1 \times 100 = 100$
$2 \times 1 = 2$	$2 \times 10 = 20$	$2 \times 100 = 200$
$3 \times 1 = 3$	$3 \times 10 = 30$	$3 \times 100 = 300$
.	.	.
.	.	.
.	.	.
$9 \times 1 = 9$	$9 \times 10 = 90$	$9 \times 100 = 900$
$1 \div 1 = 1$	$10 \div 10 = 1$	$100 \div 100 = 1$
$2 \div 1 = 2$	$20 \div 10 = 2$	$200 \div 100 = 2$
$3 \div 1 = 3$	$30 \div 10 = 3$	$300 \div 100 = 3$
.	.	.
.	.	.
.	.	.
$9 \div 1 = 9$	$90 \div 10 = 9$	$900 \div 100 = 9$

$2 \times 1 = 2$	$2 \times 10 = 20$	$2 \times 100 = 200$
$2 \times 2 = 4$	$2 \times 20 = 40$	$2 \times 200 = 400$
$2 \times 3 = 6$	$2 \times 30 = 60$	$2 \times 300 = 600$
.	.	.
.	.	.
.	.	.
$2 \times 9 = 18$	$2 \times 90 = 180$	$2 \times 900 = 1800$
$2 \div 1 = 2$	$20 \div 10 = 2$	$200 \div 100 = 2$
$4 \div 2 = 2$	$40 \div 20 = 2$	$400 \div 200 = 2$
$6 \div 3 = 2$	$60 \div 30 = 2$	$600 \div 300 = 2$
.	.	.
.	.	.
.	.	.
$18 \div 9 = 2$	$180 \div 90 = 2$	$1800 \div 900 = 2$

$$\begin{array}{lll}
2 \div 2 = 1 & 20 \div 2 = 10 & 200 \div 2 = 100 \\
4 \div 2 = 2 & 40 \div 2 = 20 & 400 \div 2 = 200 \\
6 \div 2 = 3 & 60 \div 2 = 30 & 600 \div 2 = 300 \\
\vdots & \vdots & \vdots \\
18 \div 2 = 9 & 180 \div 2 = 90 & 1800 \div 2 = 900
\end{array}$$

Careful development and discussion of these and other patterns help children learn skills for further work with the algorithm. During discussions, note relationships between the sentences in a line, for example, $1 \times 1 = 1$, $1 \times 10 = 10$, and $1 \times 100 = 100$. Also note relationships between a given line of multiplication sentences and the corresponding line of division sentences, such as $2 \times 9 = 18$, $2 \times 90 = 180$, $2 \times 900 = 1800$, and $18 \div 9 = 2$, $180 \div 90 = 2$, $1800 \div 900 = 2$.

You can prepare work sheets that contain multiplication and division patterns like these for children to complete. Discuss their completed work sheets with them to highlight the relationships. Later use oral questions to reinforce written work: "If there are 3 nines in 27, how many nines are there in 270?" "If 6 eights are 48, what are 6 eighties?"

Children's next work is usually with division that yields a quotient greater than 10 without regrouping. In the activity that follows, $48 \div 4 = 12$ is the example. The dialogue given here suggests the types of questions you should ask to help children develop a good understanding of the process.

ACTIVITY 10-15

Stamp Display

"John has 48 stamps that he is going to fix for a school display. He will put 4 stamps on a card. How many cards will he need?"

Write an algorithm for this situation:

$$4\overline{)48}$$

Use a dialogue similar to the following to help the children understand the algorithm:

Teacher: "What does the 48 stand for?"
Children: "The number of stamps John has."
Teacher: "What does the 4 stand for?"
Children: "The number of stamps for each card."
Teacher: "Does John need at least 10 cards?"

Children: "Yes."
Teacher: "How do you know?"
Children: "Ten cards will hold 40 stamps; he has more than 40 stamps."
Teacher: "Will he need at least 20 cards?"
Children: "No, 20 cards hold 80 stamps; he doesn't have that many."
Teacher: "After he puts 4 stamps on each of 10 cards, how many stamps are left?"
Children: "Eight."
Teacher: "How many cards will he need for 8 stamps?"
Children: "Two."
Teacher: "How many cards does he need for the 48 stamps?"
Children: "Twelve."

During the discussion, complete the algorithm step by step:

$$\begin{array}{r|l} 4\overline{)48} & \\ 40 & 10 \times 4 \\ \hline 8 & \\ 8 & 2 \times 4 \\ \hline \end{array}$$

Repeat with similar examples.

DIVISION WITH REGROUPING

Regrouping is required when the number in the tens place of the dividend is not a multiple of the divisor.

Concrete-Manipulative Model

Division for an example like $3\overline{)45}$ requires regrouping because 40 (4 tens) is not a multiple of 3. This example can be worked for a partitive situation with beansticks. Begin by representing 45 with the sticks, as in Figure 10-17(a). Show the regrouping as a series of steps, relating action with the beansticks to the algorithm in the margin. First, consider the tens sticks; separate them into 3 groups, each containing 1 stick (b). It is not possible to separate the remaining stick without exchanging it for 10 loose beans. This exchange is shown in (c). Finally, the 15 loose beans can be partitioned into 3 equal-sized sets by putting 5 beans alongside each tens stick. The answer is determined by adding 10 and 5 in the partial quotients.

$$3\overline{)45}$$

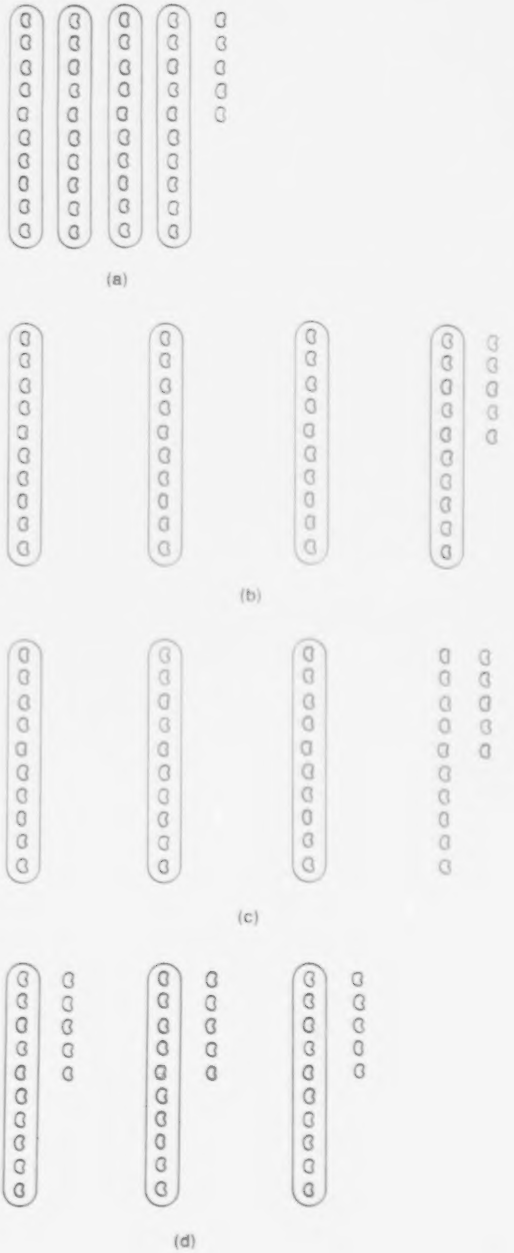
$$\begin{array}{r|l} 3\overline{)45} & \\ 30 & 3 \times 10 \\ \hline \end{array}$$

$$\begin{array}{r|l} 3\overline{)45} & \\ 30 & 3 \times 10 \\ \hline 15 & \end{array}$$

$$\begin{array}{r|l} 3\overline{)45} & \\ 30 & 3 \times 10 \\ \hline 15 & \\ 15 & 3 \times 5 \\ \hline \end{array}$$

FIGURE 10-17

Division problem 45 ÷ 3 = 15 represented with beansticks



Algorithm Form

The algorithm form should be studied next. The form shown previously, or some variation of it, is frequently used. This form is beneficial because:

1. The parts of the quotient are shown in their complete form; the 10 is shown as 10 at the right, rather than as a 1 in the tens place above the dividend.
2. The multiplication is shown in a familiar form, 3 × 10, rather than in the abbreviated form shown here.

$$\begin{array}{r} 1\\ 3\overline{)45}\\ 3 \end{array}$$

   in which the 1 indicates a 10 that is multiplied by 3, the divisor, resulting in a product of 30, which is indicated by a 3 written in the tens place beneath the dividend.

3. The product of 3 and 10 is shown as 30 rather than as 3 in the tens place.

Thus the form of the algorithm helps children see the meaning of the steps performed, because the different parts of the algorithm are related to the actions with the beansticks used earlier.

A sufficient number of examples such as the following help make the meaning of the algorithm clear as the children move from the use of manipulative materials to the abstract algorithm:

(a) $$\begin{array}{r|l} 4\overline{)56} & \\ \underline{40} & \underline{10 \times 4} \\ 16 & \end{array}$$

(b) $$\begin{array}{r|l} 4\overline{)56} & \\ \underline{40} & \underline{10 \times 4} \\ 16 & \\ \underline{16} & \underline{4 \times 4} \\ & 14 \end{array}$$

Teacher: "When we divide 56 by 4, will there be as many as 10 fours in 56?"

Children: "Yes, because 10 × 4 = 40; 40 is less than 56."

Teacher: "Will there be as many as 20 fours in 56?"

Children: "No, 20 × 4 = 80; 80 is more than 56."

Teacher: "Let us show [in (a) in the margin] that 10 times 4 equals 40 and then subtract 40 from 56 to see how much is left to be divided. What is 56 minus 40?"

Children: "Sixteen."

Teacher: "What is 16 divided by 4?"

Children: "Four."

Teacher: "We can show that too [in margin example (b)]. Now what is the answer to 56 divided by 4?"

Children: "Fourteen."

DIVISION INVOLVING LARGER NUMBERS

A good understanding of division with quotients represented by 1-digit numerals, including use of the algorithm and the ability to multiply by 10 and powers of 10, is a prerequisite to learning division involving larger numbers. Skill in estimating quotients is developed through activities like those that follow.

Two-digit Divisors and One-digit Quotients

Children's first work with 2-digit divisors is usually done with divisors that are multiples of 10. This is so because patterns can be used to help them develop skill in determining quotients. The patterns illustrated earlier in this chapter lead to this generalization:

____ones × ____tens = ____tens
____tens ÷ ____tens = ____ones

So, for 270 ÷ 30 = ☐, the 27 tens are divided by 3 tens to determine the quotient, 9.

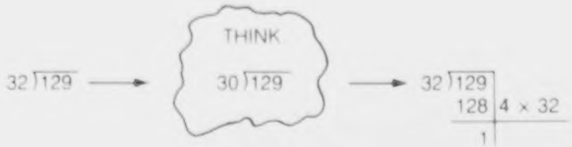
Answers for examples such as these, which have remainders,

$30\overline{)123}$ $\quad 30\overline{)98}$ $\quad 30\overline{)72}$ $\quad 30\overline{)131}$

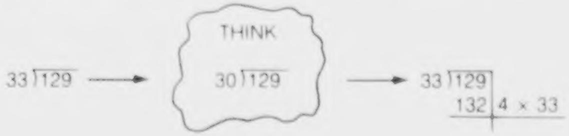
can be determined by using this multiplication pattern:

$1 \times 30 = 30 \qquad 2 \times 30 = 60 \qquad 3 \times 30 = 90$
$4 \times 30 = 120 \qquad 5 \times 30 = 150$

Because not all divisors are multiples of 10, strategies for estimating quotients for nonmultiples must be learned. A useful strategy is to round the divisor to the nearer 10 and use multiples of that number:



This process will not always yield the correct quotient, as in



so children must be able to adjust the quotient to the correct one by using a smaller number, in this case 3.

Two-digit and Larger Quotients

Problem situations that enable children to visualize the meaning of each term in the algorithm are useful for developing an understanding of division involving larger numbers. Examples like the following should be used.

ACTIVITY 10-16

Bagging Marbles

Present a problem situation: "288 marbles are to be packaged so that there are the same number of marbles in each of 24 bags. How many marbles will be in each bag?"

Write the division sentence for this problem and an algorithm:

$$288 \div 24 = \square \qquad 24\overline{)288}$$

Use a dialogue similar to this one to help children solve the problem and learn the algorithm.

Teacher: "What does the 288 stand for?"
Children: "The total number of marbles."
Teacher: "What does the 24 stand for?"
Children: "The number of bags into which marbles will be put."
Teacher: "Will there be at least 10 marbles in each bag?"
Children: "Yes, because 24 times 10 equals 240 and there are more than 240 marbles."
Teacher: "Will there be at least 20 marbles in each bag?"
Children: "No, because 24 times 20 is 480 and there are not that many marbles."

Show the result of putting 10 marbles in each bag in the algorithm:

$$\begin{array}{r|l} 24\overline{)288} & \\ \underline{240} & \underline{24 \times 10} \\ & \end{array}$$

Continue the dialogue:

Teacher: "After 10 marbles have been put in each bag, how many marbles are left to be packaged?"
Children: "48." (Help the children determine this response, if necessary.)

- Show the algorithm with the remainder:

$$\begin{array}{r|l} 24\overline{)288} & \\ \underline{240} & \underline{24 \times 10} \\ 48 & \end{array}$$

- Continue the dialogue:

Teacher: "Are there enough marbles to put 1 more in each bag?"
Children: "Yes, because that would use up only 24 of the 48 marbles."
Teacher: "Can we put 2 more in each bag?"
Children: "Yes, because 24 times 2 equals 48; we have that many marbles."
Teacher: "We have used the 48 marbles. I'll show that in the algorithm."

$$\begin{array}{r|l} 24\overline{)288} & \\ \underline{240} & \underline{24 \times 10} \\ 48 & \\ \underline{48} & \underline{24 \times 2} \end{array}$$

Teacher: "When we add the 10 and 2, we find that there will be 12 marbles in each bag. So 288 divided by 24 equals 12."

- Before you leave the activity, relate each numeral in the algorithm to the problem situation so that children understand each one's meaning. Ask questions like, "What does this numeral stand for?" as you point to the 240. (It represents the number of marbles that have been bagged after 10 marbles have been put in each bag.)
- Repeat with similar problems until the algorithm is clear to the children.

If children have a good understanding of the meaning of division, know how to multiply by tens and powers of 10, and know how to use the algorithm with smaller numbers, the first work they do with 2-place and larger divisors need not be restricted to examples that have 1-place quotients. In the foregoing example, the quotient is larger than 10. Further use of examples with 1- and 2-digit numerals to represent quotients smaller than 100 will further develop understanding of the algorithm.

Estimating Quotients

Skills in determining the quotient must be continuously refined as numbers increase in size. The following strategy can be used.

ACTIVITY 10-17 Estimating Quotients

- Begin with an algorithm:

$$36\overline{)7348}$$

- Ask questions similar to these:

Teacher: "Are there at least 10 thirty-sixes in 7348?"
Children: "Yes. Ten times 36 is 360; that is less than 7348."
Teacher: "Are there as many as 100 thirty-sixes in 7348?"
Children: "Yes. One hundred times 36 is 3600; that is less than 7348."
Teacher: "Are there as many as 1000 thirty-sixes in 7348?"
Children: "No. One thousand times 36 is 36,000; that is too much."
Teacher: "We know that the answer is more than 100 but less than 1000. Will there be as many as 200 thirty-sixes in 7348?"
Children: "Yes. Two hundred times 36 is 7200; that is less than 7348."
Teacher: "Can there be as many as 300 thirty-sixes?"
Children: "No, that is too many."

- Show this step in the algorithm:

$$\begin{array}{r|l} 36\overline{)7348} & \\ \underline{7200} & \underline{200 \times 36} \\ 148 & \end{array}$$

- Continue the dialogue:

Teacher: "After we multiply 36 by 200 and subtract, we have a remainder of 148. How many 36s in 148?"
Children: "There are 4."

- Complete the algorithm:

$$\begin{array}{r|l} 36\overline{)7348} & \\ \underline{7200} & \underline{200 \times 36} \\ 148 & \\ \underline{144} & \underline{4 \times 36} \\ 4 & 204 \end{array}$$

- Review the meaning of each term in the algorithm, including the undivided remainder, 4.
- Repeat with similar examples.

Shortened Forms

Refinements can be made in the division process so that more mature ways are used with the algorithm to reduce the number of steps and lessen the notation that is written as division is done. For example, to help children decide the number of decimal places there will be in a quotient, place a number of division examples on the chalkboard:

$$39\overline{)4869} \qquad 73\overline{)7421} \qquad 86\overline{)6243} \qquad 23\overline{)78{,}552}$$

The place value of each quotient figure is revealed by multiplying the divisor by 10 and its powers. In the first example, this shows that the answer is more than 100 but less than 1000, so there will be a quotient represented by a 3-digit numeral. An × or something similar can be marked over the place-value position, as shown in the margin. Next, the first quotient figure should be calculated. Because in this example it is known to be some number of hundreds, the exact number of hundreds is found by multiplying by multiples of 100. Once children can readily discover the first quotient figure, they can begin to shorten the steps used with the algorithm. First, the number of decimal places is decided and marked. Next, the first quotient figure is determined. The number of 39s in 4869 is estimated by rounding 39 to 40 and multiplying 40 by multiples of 100. It is more than 100 but less than 200, so "100" is placed in the algorithm, as shown in (a) in the margin. The numeral for the product of 39 and 100 is placed beneath the dividend in the algorithm. The number of 39s in 969 will be about 20 (39 can be rounded to 40 to make the estimate), so "20" is written in the algorithm, as in (b). The product of 20 and 39 is subtracted from 969. Finally, the number of 39s in 189 is estimated by rounding off to 40. The answer, "4," is written in the algorithm, and the numeral for the product of 4 and 39 is written beneath 189. The remainder is then determined. The completed algorithm form is shown in (c).

$$\begin{array}{r} \times \\ 39\overline{)4869} \end{array}$$

$$\begin{array}{rr} & \times \\ & 100 \\ \text{(a)} & 39\overline{)4869} \\ & \underline{3900} \\ & 969 \end{array}$$

$$\begin{array}{rr} & \times 20 \\ & 100 \\ \text{(b)} & 39\overline{)4869} \\ & \underline{3900} \\ & 969 \\ & \underline{780} \\ & 189 \end{array}$$

$$\begin{array}{rr} & 4 \\ & \times 20 \\ & 100 \\ \text{(c)} & 39\overline{)4869} \\ & \underline{3900} \\ & 969 \\ & \underline{780} \\ & 189 \\ & \underline{156} \\ & 33 \end{array}$$

Some children will continue to reduce the number of written steps until they put just a "1" in the hundreds place, a "2" in the tens place, and a "4" in the ones place, rather than writing "100," "20," and "4." These children may or may not shorten the way in which products of the quotient figures and divisor are written or the way in which the subtraction is done. The only reason zeros might be omitted in the algorithm is to save time. However, there is an increased likelihood that mistakes will be made when zeros are omitted, so do not have children leave them out when they are not ready for this step.

Some children will achieve an understanding of the most mature form of the algorithm, explained for the example in the margin. The divisor, 32, is rounded to 30 to give an easier-to-use divisor. The number of 30s in 79 is 2. A "2" is recorded in the hundreds place of the quotient because 79 stands for hundreds. The multiplication of 2 and 32 is completed, and the product, 64, is recorded. Sixty-four is subtracted from 79, and the answer, 15, is

$$\begin{array}{r} 248\text{r}10 \\ 32\overline{)7946} \\ \underline{64} \\ 154 \\ \underline{128} \\ 266 \\ \underline{256} \\ 10 \end{array}$$

recorded; the 4 is brought down. Because 32 has been rounded to 30, the second division (that is, 154 divided by 30) gives a quotient of 5. Five is too large, however, and the quotient is reduced to 4. The division is completed by multiplying 4 times 32, recording the 128, subtracting, bringing down the 6, and dividing again to find the number of ones in the quotient. The answer is completed by writing the remainder after the quotient.

Many children who understand the meanings of multiplication and division and who can determine which operation to use when confronted with a problem situation, experience considerable difficulty while learning to use an algorithm for 1 or both of the operations. Rather than present them with pages of tedious and time-consuming practice in an effort to improve their skills, it is better that they learn to compute answers on a calculator. Time spent on paper-and-pencil practice exercises frequently results in few if any long-lasting results. This time is better spent on other topics in mathematics.

HANDLING REMAINDERS IN DIVISION

When the divisor is not a whole number factor of a dividend, a remainder occurs as the division is completed. Children should learn to treat remainders according to the nature of the situations giving rise to division. The following situations illustrate why division with remainders needs special attention in the intermediate grades:

"There are 32 children in our class. If we play a game that requires 3 equal-sized teams, how many players will be on each team?" A paper-and-pencil-computed answer is 10r2. If computation is done on a calculator, the display shows 10.666666. In either event, the quotient to this partitive situation is 10, with something left over. Since each team must have the same number of players, the correct response is simply 10.

"Mother baked 26 cookies for Jamelda and her 3 friends. If the cookies are distributed equally among the 4 children, how many cookies will each child get?" The paper-and-pencil answer to this partitive situation is 6r2, while the calculator answer is 6.5. Now children must see that there will be 4 groups of 6 cookies, with 2 cookies left over. The 2 cookies can be divided equally among the children, so the whole-number remainder can be written as $\frac{2}{4}$, or $\frac{1}{2}$. The calculator already shows 0.5, or $\frac{1}{2}$. The acceptable answer is $6\frac{1}{2}$.

"It costs 79 cents to buy 3 cans of cat food. What is the price for 1 can?" The paper-and-pencil answer is 26r1, while the calculator answer is 26.333333. In this case there is a whole-number quotient of 26, with something left over. Children will understand that the store is not likely to sell 1 can of cat food for 26 cents because then the price would be 3 cans for 78 cents. For this example, the answer needs to be changed to 27 when the price of 1 item is determined.

"A group of 53 children is riding to camp in station wagons. If each wagon can carry 6 children and their gear, how many station wagons will be needed?" The answers to this measurement situation are 8r5 and 8.833333. The whole-number quotient shows that 8 station wagons will be needed. However, the remainder shows that there are children without room in the 8 wagons; 1 additional wagon is needed to transport the 5 children, so the answer is 9.

"Sixty-eight apples must be put in boxes. If each box holds 8 apples, how many full boxes are there?" For this measurement situation, the paper-and-pencil answer is 8r4; the calculator answer is 8.5. There are enough apples to fill 8 boxes, with 4 apples or 0.5 of another box left unboxed, so the answer is 8.

Starting with late third grade and continuing through the remainder of the elementary-school years, children should have realistic exercises that require them to interpret how remainders should be treated. There are 2 ways in which remainders are typically handled as practice pages of algorithms are completed. In example (a) in the margin, the remainder is left intact at the bottom of the algorithm. It is also correct to write it after the quotient, using the letter *r* to denote that it is a remainder (b). The form in (c), where the remainder is recorded as a common fraction, is done less frequently today than in the past.

(a) $$\begin{array}{r} 7 \\ 9\overline{)67} \\ \underline{63} \\ 4 \end{array}$$

(b) $$\begin{array}{r} 7r4 \\ 9\overline{)67} \\ \underline{63} \\ 4 \end{array}$$

(c) $$\begin{array}{r} 7\tfrac{4}{9} \\ 9\overline{)67} \\ \underline{63} \\ 4 \end{array}$$

Summary

There are three interpretations of multiplication with which children should become acquainted. Each one can be demonstrated with manipulative materials and story situations to make them meaningful. These interpretations are repeated addition, the array, and Cartesian product. Repeated addition is used most often to introduce multiplication because it can be related to the already understood addition operation. Arrays are easily represented by objects such as poker chips and dot patterns and are useful for showing basic multiplication combinations that involve a given factor, such as all combinations with a factor of 7. Cartesian products result from situations in which all of the ordered pairs that result from matching each element in 1 set with each element in another set are determined.

Two types of situations give rise to division, measurement and partitive. Children should be introduced to both types of situations through meaningful real-life problems and manipulative-material activities. Strategies for teaching basic facts and helping children learn them include use of the multiplication table, a special characteristic of multiplying by 9, and relating divisors to multiplication with practice materials.

The basic properties of multiplication—commutativity, associativity, distribution of multiplication over addition, and the roles of 1 and 0—can be presented through activities with markers, arrays, multiplication tables, number lines, blocks, and other manipulative devices.

Care must be used when children are taught to use algorithms for multiplication and division so the algorithms' meanings and applications will be clear. Materials to use include place-value devices, markers for magnetic or flannel boards, arrays, and structured materials. Real-life story problems play an important role in making applications of the algorithms meaningful. Intermediate-grade children should learn to use a calculator to compute multiplication and division answers. Division situations frequently result in answers with remainders other than 0. Children should learn to handle remainders according to the nature of the situation, rather than by rule alone.

Study Questions and Activities

1. A process of factoring numbers is suggested as one means of helping children learn the basic multiplication facts (see pp. 366–367). Demonstrate your understanding of this process by explaining how a child might think about these multiplication combinations:

   $6 \times 9 = \square \quad 8 \times 7 = \square \quad 9 \times 8 = \square$

2. Dialogues are used in this chapter to illustrate how you can help children develop an understanding of the division algorithm. Develop a hypothetical dialogue for each of these sentences: $216 \div 12 = 18$ and $645 \div 15 = 43$. First, write a story problem that portrays a measurement situation, then one that portrays a partitive situation for each sentence. Identify problems that are measurement and those that are partitive. Write a dialogue similar to the one in Activity 10-15 for one of your measurement situations and a dialogue similar to the one in Activity 10-16 for a partitive situation. Include the step-by-step process for completing the algorithm for each situation.
3. Five different ways of handling remainders in division are explained in this chapter. Write 1 word problem for each type of situation. Explain how the remainder should be handled for each of your problems.
4. Broadbent describes the lattice method of multiplying and dividing in an article in the *Arithmetic Teacher* (see For Further Reading section). Read the article and use the algorithms to compute the answer for each of these sentences: $246 \times 195 = \square$ and $1176 \div 42 = \square$. Evaluate each algorithm in terms of its clarity and ease of use. Under what circumstances might you use these algorithms with children?

For Further Reading

Adkins, Bryce E. "A Rationale for Duplication-Mediation Multiplying," *Arithmetic Teacher*, XI, No. 4 (April 1964), pp. 251–253. Explains an old multiplication process, frequently called Russian-peasant multiplication. Some students will enjoy using this procedure to supplement the regular multiplication algorithm.

Ando, Masue, and Hitoshi Ikeda. "Learning Multiplication Facts—More than Drill," *Arithmetic Teacher*, XVIII, No. 6 (October 1971), pp. 359–364. Presents activities that develop understanding of the basic multiplication facts, ways of organizing the facts in tables, and procedures for memorizing them.

Ashlock, Robert. "Model Switching: A Consideration in the Teaching of Subtraction and Division of Whole Numbers," *School Science and Mathematics*, LXXVII, No. 4 (April 1977), pp. 327–335. Models are used to help children understand operations and their algorithms. Chips or blocks may be used to represent partitive and measurement division situations. Ashlock cautions teachers to use models consistently so children do not become confused by discrepancies between how a model is used and the problem situation it represents.

Bolduc, Elroy J., Jr. "The Monsters in Multiplication," *Arithmetic Teacher*, XXVIII, No. 3 (November 1980), pp. 24–26. Ways to help children master the basic multiplication facts are discussed. Finger multiplication and the times table are suggested as two worthwhile activities.

Broadbent, Frank W. "Lattice Multiplication and Division," *Arithmetic Teacher*, XXXIV, No. 5 (January 1987), pp. 28–31. The lattice algorithm was used in Europe as early as the 1400s. A modern version of the algorithm for both multiplication and division is explained in this article. The author suggests that the lattice algorithm might substitute for the standard algorithm during instruction about multiplication and division.

Bruni, James V., and Helene J. Silverman. "The Multiplication Facts: Once More, With Understanding," *Arithmetic Teacher*, XXIII, No. 6 (October 1976), pp. 402–409. Several activities and games using arrays provide the means for developing understanding and memorization of the basic multiplication facts.

Cacha, Frances B. "Exploring the Multiplication Table and Beyond," *Arithmetic Teacher*, XXVI, No. 3 (November 1978), pp. 46–48. Close study of the multiplication table reveals the properties of multiplication, information about odd and even factors, patterns on diagonals, and digit sums, all of which help children understand and master the facts.

———. "Understanding Multiplication and Division of Multidigit Numbers," *Arithmetic Teacher*, XIX, No. 5 (May 1972), pp. 349–354. Illustrates the use of arrays to interpret the meanings of the multiplication and division algorithms. The illustrations can serve as models for transparencies or magnetic board manipulatives.

Cheek, Helen Neely, and Melfried Olsen. "A Den of Thieves Investigates Division," *Arithmetic Teacher*, XXXIII, No. 9 (May 1986), pp. 34–35. A story of a group of thieves who have taken a large sum of money serves as the scenerio for helping children learn division with large numbers. The process is similar to that recommended in this chapter.

Dunkels, Andrejs. "More Popsicle-Stick Multiplication," *Arithmetic Teacher*, XXIX, No. 7 (March 1982), pp. 20–21. Common popsicle sticks are models for multiplying numbers between 10 and 100. Step-by-step instructions are explained and illustrated.

Falco, Thomas. "Multiplication Clue: Game Activity for the Classroom," *Arithmetic Teacher*, XXXI, No. 2 (October 1983), pp. 36–37. Through dialogue and chalkboard examples, the author explains a game that requires children to analyze the teacher's responses to determine factors of numbers. The game is a good one for helping children reinforce learning of the multiplication facts.

Hazekamp, Donald W. "Teaching Multiplication and Division Algorithms," in *Developing Computational Skills*, 1978 Yearbook, National Council of Teachers of Mathematics (Reston, Va.: The Council), pp. 96–128. A variety of manipulative materials—abacus, beansticks, base-ten blocks, and others—serves as models for developing understanding of the algorithms. Many of the activities are illustrated.

Junge, Charlotte W. "Now Try This—In Multiplication," *Arithmetic Teacher*, XIV, No. 2 (February 1967), pp. 134–135. Explains several shortcuts used to multiply certain types of whole numbers. Gives 3 procedures for multiplying a pair of numbers ending in 5.

Laing, Robert A., and Ruth Ann Meyer. "Transitional Division Algorithms," *Arithmetic Teacher*, XXIX, No. 9 (May 1982), pp. 10–12. These authors contend that there is a higher percentage of children failing to understand division algorithms today than in the past. They believe that the use of transitional algorithms is to blame and that use of such algorithms should be curtailed in favor of a more direct approach.

Meyer, Ruth A., and James E. Riley. "Multiplication Games," *Arithmetic Teacher*, XXXIII, No. 8 (April 1986), pp. 22–25. Five multiplication games with different levels of difficulty are illustrated and described. Each one deals with multiplication facts.

Robold, Alice I. "Grid Arrays for Multiplication," *Arithmetic Teacher*, XXX, No. 5 (January 1983), pp. 14–17. Arrays are commonly used to introduce multiplication. Ways to use them with larger numbers are described in this article.

Smith, C. Winston, Jr. "Tiger-bite Cards and Blank Arrays," *Arithmetic Teacher*, XXI, No. 8 (December 1974), pp. 679–682. Arrays with "bites" taken from them serve as a basis for children's searches for missing factors when the total (product) and number of rows (given factor) are known. The cards lead to a useful way of investigating the meaning of the division algorithm.

Souviney, Randall J. "Giving Division Some Meaning," *Learning*, V, No. 6 (February 1977), pp. 68–69. The 2 types of division situations are described and illustrated. Representational materials, in the form of tens strips and small squares, are used to help children understand the meaning of the algorithm for each type of division situation.

Stuart, Maureen, and Barbara Bestgen. "Productive Pieces: Exploring Multiplication on the Overhead," *Arithmetic Teacher*, XXIX, No. 5 (January 1982), pp. 22–23. An overhead projector, a transparent grid, and colored transparent rectangles that fit the grid serve to develop children's understanding of multipli-

cation. The materials make a nice tie-in of semiconcrete materials and the abstract multiplication table.

Zweng, Marilyn J. "Division Problems and the Concept of Rate," *Arithmetic Teacher*, XI, No. 8 (December 1964), pp. 547–556. This is a report on research done with second graders to determine which types of division situations are easiest for them to understand. Zweng found that measurement-rate situations were the most easily understood.

———. "The Fourth Operation Is Not Fundamental," *Arithmetic Teacher*, XIX, No. 8 (December 1972), pp. 623–627. This article contains a discussion of division. It has good examples of partitive and measurement situations, with sample illustrations.

Chapter

11

Fractional Numbers—Common Fractions

In Chapter 11, you will read about

1. five meanings for common fractions in everyday life
2. the importance of understanding common fractions through concrete materials for representing parts of regions and sets
3. a sequence of activities emphasizing equivalent fractions and comparison of unlike fractions
4. strategies for naming or renaming equivalent fractions, including prime factorization
5. concrete models for demonstrating addition, subtraction, multiplication, and division with common fractions and understanding these operations
6. development of ideas behind cancellation and invert-and-multiply processes
7. activities emphasizing ratio and proportion
8. common problems children have with fractions and some suggestions for overcoming these problems

Children's study of fractional numbers begins as early as kindergarten and continues through elementary school. The fractional numbers treated during the elementary-school period are a part of the set of rational numbers; those that can be expressed in the form *a*/*b*, when *a* is any whole number and *b* is any nonzero whole number. It is possible to consider such numbers on a purely abstract basis with the meanings of the numbers and operations developed from undefined terms, postulates or axioms, and the theorems that pertain to them. However, such an approach is not meaningful to elementary-school children, so a concerete approach to the study of fractional numbers must be taken.

The most useful approach is one that allows children to intuitively develop a reasoned understanding of fractional numbers. In all grades of elementary school, many concrete representations of fractional numbers help children develop a clear understanding of the meaning of these numbers and the way operations with them are performed.

Fractional numbers, like whole numbers, can be expressed in several different ways. Symbolically they may be expressed in any one of 3 ways: as common fractions ($1/2$ and $2/3$), as decimal fractions (0.5 and 0.6666), and as percent (50 percent and $66\frac{2}{3}$ percent). Also, any given fractional number has an infinite number of numerals, for example, $1/2 = 2/4 = 3/6 = 4/8 = 5/10 = \ldots$

THE MEANING OF COMMON FRACTIONS

Common fractions are numerals used to represent fractional numbers and ratios. Historically children have experienced considerable difficulty with the meaning of common fractions because there are several situations out of which they arise, and these have not always been well understood. If they are to understand common fractions, children must become familiar with each situation and work with it in a mature fashion.

Unit Subdivided Into Equal-Sized Parts

When a unit of measure, such as an inch, is subdivided into equal-sized parts, common fractions can be used to express the meaning of each subunit. For example, when an inch is subdivided into 2 equal-sized parts, each part is $1/2$ of an inch. Likewise, when an object such as a cake is first considered as a whole and is then cut into 4 parts of equal size, the common fraction $1/4$ can be used to express the size of each part. The parts of a common fraction indicate the nature of the situation from which the numeral arises. In the numeral $\frac{1}{2}$, the 2 indicates the number of equal-sized parts into which the whole, or unit, has been subdivided. This part of the fraction is the denominator. The 1 indicates the number of parts being considered at the moment and is called the numerator.

2 Set Subdivided Into Equal-Sized Groups

Situations that lead to fractional numbers when a set is subdivided into groups of equal size are clearly related to those that involve division. When a set of 12 objects is subdivided into 2 equal-sized groups, the mathematical sentence $12 \div 2 = 6$ represents the situation. The 6 in the sentence represents $\frac{1}{2}$ of the original set of 12. The sentence $\frac{1}{2}$ of $12 = 6$ also describes this situation. Children will often engage in activities that involve parts of sets of discrete wholes. To find $\frac{1}{6}$ of 18, for instance, children must think of 18 objects that are to be subdivided into 6 groups of equal size. The size of each group relates to the size of the original set in such a way that each is $\frac{1}{6}$ of the original set. When a common fraction is used to represent this type of situation, the denominator indicates the number of equal-sized groups into which the set is subdivided, and the numerator indicates how many of the groups are being considered.

3 Expression of Ratios

The relationship between a pair of numbers is often expressed as a ratio. Ratios arise from many situations. Examples of common situations are:

- A comparison made between the number of children in a classroom and the number of textbooks in the same classroom. If each child has 6 textbooks, the ratio is 1 to 6 and may be expressed as $\frac{1}{6}$ or 1 : 6. Here the numerals in the ratio expression represent the numbers of objects in 2 completely different sets.
- A comparison made between the number of blue-covered books in a set and the number of books in the complete set. If there are 3 blue-covered books in a set of 10 books, the ratio is 3 out of 10 and may be expressed as $\frac{3}{10}$ or 3 : 10. Here the numerals in the ratio represent the numbers of objects in a set and 1 of the groups within it.
- A comparison made between the length of 2 objects. If 4 pieces of dowel rod, each the same length, are laid end-to-end alongside a second rod and have a total length as long as the second rod, the ratio is 4 to 1 and may be expressed as $\frac{4}{1}$ or 4 : 1. (The length of one short rod is $\frac{1}{4}$ of the longer rod.) Here the numerals represent the number of shorter objects compared with a single object.

Indicated Division

Sentences such as 3 ÷ 4 = ☐ and 17 ÷ 3 = ☐ indicate that division is to be performed. A physical model of the first sentence is the subdivision of 3 feet of cloth into 4 equal-sized parts, with the answer, $\frac{3}{4}$, indicating the size of each of the 4 parts. The second sentence might represent a situation in which 17 cookies are shared equally by 3 children, with $5\frac{2}{3}$ being each child's share.

Expression of Rational Numbers

At a completely abstract level, it is possible to think of common fractions as representing the elements in the set of nonnegative rational numbers. Though children in the elementary school will not study rational numbers in this abstract sense, the development of concepts of fractional numbers and common fractions should be consistent with that of concepts of rational numbers.

INTRODUCING COMMON FRACTIONS

There are those who advocate that common fractions not be taught in elementary school. They contend that the calculator and metric system will make common fractions obsolete and that children will not need to learn about them. This view is shortsighted for at least 2 reasons.

First, the preceding discussion shows that common fractions are used in ways unrelated to those that the calculator and metric system will replace. Children need to learn these uses because their importance will not diminish. While it is true that operations with fractional numbers expressed as common fractions are not frequently done, children should be introduced to the operations in ways discussed in this book, even if they are not expected to master each algorithm.

Second, at the present time there is no reason to expect that the United States will soon become a metric nation. Until the time when it does, children will need to know how common fractions are used in the customary system of measurement.

Primary-grade children are usually introduced to simple common fractions—those with denominators of 2, 3, 4, 5, 6, and 8—and with fractional parts of groups of perhaps 20 objects. They also compare the sizes of the common fractions. Operations with common fractions, simplication (reduction), changing to common denominator, renaming in higher terms,

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least common multiple (LCM), and cancellation are commonly taught in the intermediate grades.

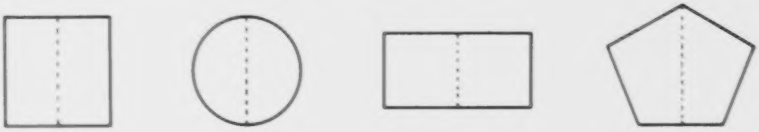
Using Regions

The study of common fractions should begin with a variety of geometric regions for children to subdivide into equal-sized parts. Develop each concept carefully so that children understand the meanings of the parts of the whole. While it is not necessary that the equal-sized parts be the same shape, it is easier for young children to understand the meanings of common fractions when the parts are congruent. The following activity describes a meaningful way to introduce first graders to the concept of a common fraction:

ACTIVITY 11-1

Geometric Regions

- Give each child several geometric regions and a pair of classroom scissors.
- Let children experiment to see how many ways they can fold and cut each region to show 2 pieces that are the same size.



- After time for exploration, have children discuss and show what they have done. Help them use the words *one-half, halves,* and *a half* to describe their parts.

Have children subdivide appropriate regions into fourths until the meaning of the subdivisions becomes clear (Figure 11-1). Later, when other fractions are introduced, the models for some regions will need to have lines on them to show where cuts are to be made. Otherwise children will have difficulty. For example, it is difficult for young children to fold and cut blank paper models for thirds and fifths. During children's early work, do not introduce numerals for common fractions or the terms *numerator* and *denominator*. These introductions should be delayed until children thoroughly understand the fraction concept.

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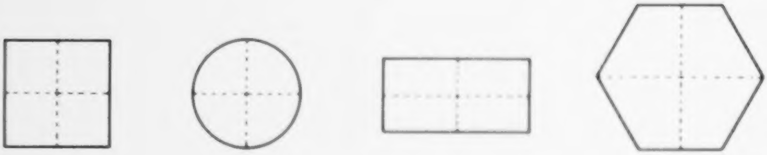


FIGURE 11-1

Folded regions showing the fraction ¼

From time to time, work sheets containing models like those in Figure 11-2 should be completed by children. In (a) they color one of the 3 parts in each region to show ⅓; in (b) they put an X on the shapes that do not show halves. Workbooks for first and second graders contain a limited number of practice exercises of these types; you can create cardboard pizzas, pies, and other "food" items for children who need further practice with the identification of simple fractional parts of a whole.

Once the meanings of the shapes and their parts are clear, introduce common fractions as numerals for fractional numbers. Relate the numerals to shapes that have been cut into 2, 3, or more parts. At first use only 1 part of each shape to show the related unit fraction. (A unit fraction is one with a numerator of 1.) Refrain still from using *denominator* and *numerator*. Do help children understand, however, that the bottom part of the numeral tells the number of equal-sized parts into which a shape has been cut, and the top part tells the number of parts being considered at the moment. Later introduce numerators greater than 1 as children consider ⅔ of a shape cut into 3 equal-sized parts, or ⅚ of a shape cut into 6 equal-sized parts.

Real-life situations provide story opportunities for introducing numerators greater than 1. "Tracy had pizza last night. Six people were expected, so the pie was cut into 6 equal parts. Only 5 people came. How much was eaten if each person ate only 1 piece?" This story can be illustrated with flannel-board pieces or drawings.

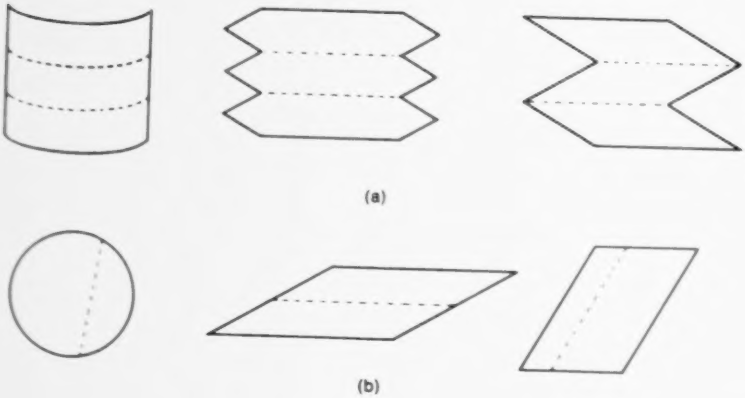


FIGURE 11-2

Samples of items to use on worksheets. (a) Color each shape to show one of the 3 equal-sized parts. (b) Cross out a shape if the 2 parts are not equal in size.

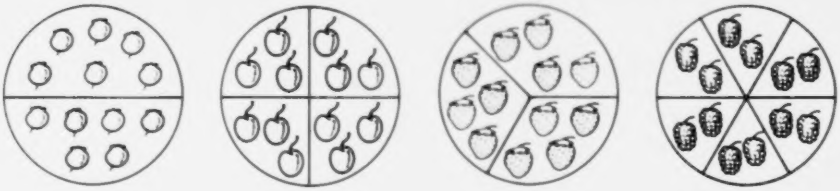
Using Sets of Objects

The concept of a fractional part of a set can be introduced only after children have a good grasp of whole numbers and are skillful in counting the number of objects in a set. Finding fractional parts of regions is often easier for children than recognizing fractional parts of sets. The association of new work with sets to previous work with regions is important. Activity 11-2 shows one way to relate understanding of parts of a set to fractional regions.

ACTIVITY 11-2

Berry Pie

- Cut several circles from construction paper or tagboard and mark them into halves, thirds, fourths, and sixths. These are the pies.
- Children can cut out shapes to represent raspberries, cherries, strawberries, and blueberries. Have them count out sets of 24 of each berry.
- Explain that the pies have been cut into equal parts and that you want to make sure that each person will receive an equal number of berries in each piece of pie. Have each child take a set of 12 berries and place them on a pie so that an equal number of berries are in each region.



- Describe what is happening as the child completes each pie. "The pie has 6 parts. There are 12 berries. Two berries are on each piece of pie. One-sixth of 12 berries is 2."
- Separate sets of 6, 18, and 24 berries on the pie with 6 pieces.
- Let children describe what they are doing and the results of dividing the sets into equal parts.
- Repeat with pies cut into 4, 5, or 8 pieces and sets of berries that can be equally divided among them.
- Variation: After the idea is well established, ask children what they would do if the pie had 4 parts and 22 berries.

Children should soon be able to divide sets into equal groups without having the fractional regions as a cue. Give each child 8 objects and ask for 2 equal-sized groups, then 4 equal-sized groups. Repeat with other sets that can be divided into 2, 3, 4, or 5 equal-sized groups until children understand. Children may notice that some sets cannot be divided into equal-sized parts with a whole-number answer for each subset. This realization sets the stage for operations for fractional numbers, discussed later in this chapter.

COMPARING FRACTIONAL NUMBERS

As children study whole numbers, they learn to compare numbers so that they can tell when 1 number is greater than, less than, or equal to another. They use sets, number lines, and other means of comparing numbers until the idea is clear. They learn that 1 number is greater than another if it is to the right of the other on the number line. At the same time, they learn that every whole number has an immediate predecessor and an immediate successor. It is always possible to determine the predecessor of a whole number greater than 0 by subtracting 1 from it. The successor of a number is determined by adding 1 to it. As children study fractional numbers, they learn that these numbers can be ordered by size, even though there is no constant value between fractional numbers as there is between every pair of adjacent whole numbers. Children also learn that a fractional number has no immediate predecessor or successor. An infinite number of fractional numbers lie between any pair, so fractional numbers do not have immediate predecessors or successors. This can be demonstrated by the fact that it is always possible to determine a fractional number midway between a pair of fractional numbers. The number midway between $5/8$ and $3/4$ is $11/16$; it is determined by expressing $3/4$ as $6/8$ and adding it to $5/8$; $5/8 + 6/8 = 11/8$ and $11/8 \div 2 = 11/16$.

It is possible to compare the sizes of 2 fractional numbers by multiplying the numerator of the first by the denominator of the second, and the denominator of the first by the numerator of the second, and comparing the 2 products. For the 2 fractional numbers a/b and c/d, when $a \times d = b \times c$, the 2 numbers are equivalent. The fractional number a/b is greater than c/d when $a \times d > b \times c$; the fractional number a/b is less than c/d when $a \times d < b \times c$.

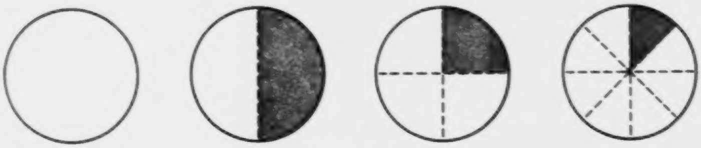
It should be clear that such abstract approaches for ordering and comparing numbers are unsuitable for children. Initial experiences should come through investigations with models of various kinds. Later, abstract procedures can be used with mathematically mature children in the higher grades of the elementary school.

1 Using Congruent Geometric Shapes

Activity 11-3 uses materials that are appropriate for children in grades three and four to compare common fractions.

ACTIVITY 11-3 Congruent Shapes

- Give each child paper cut in circles, squares, or other shapes that they can fold and cut to show halves, fourths, and eighths.
- Children fold 1 piece to show halves, cut it, and color ½ with a crayon. One piece is folded and cut to show fourths, and another is folded and cut to show eighths. One fourth piece and 1 eighth piece are colored. One unit piece is neither cut nor colored.

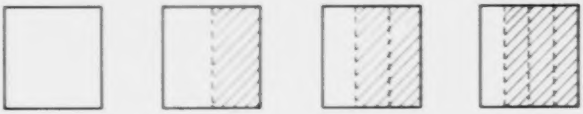


- Have children compare the colored pieces to see that ½ > ¼ > ⅛.
- Shapes cut and colored to show thirds, sixths, ninths, and twelfths, and halves, fifths, and tenths should be used to compare those fractions.

Congruent shapes are also useful for investigations in which children compare nonunit fractions. Models folded and shaded like those in Figure 11-3 help them see that ¾ > ⅔ > ½. As long as the shapes are congruent, children can see and order any fractional numbers that can be conveniently represented with shapes.

FIGURE 11-3

Geometric shapes used to compare the size of ½, ⅔, and ¾

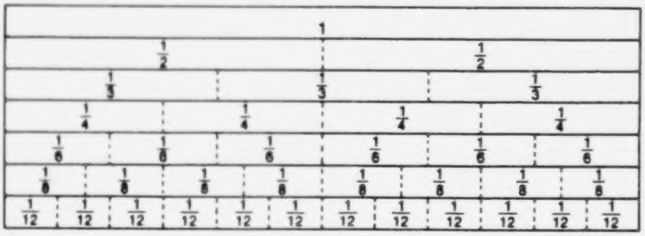


2 Using Fraction Strips

Fraction strips are also useful for comparing fractional numbers. Cut a set of strips from colored cardboard and attach magnets on the back of each strip for manipulation on a magnetic board. One useful set is pictured in Activity 11-4.

ACTIVITY 11-4 Fraction Strips

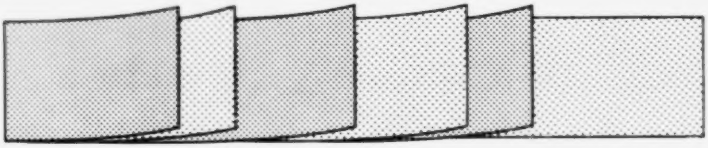
Arrange the set of strips on the magnetic board.



Ask questions like these:

- ☐ How many ½ strips are as long as the 1 strip? How many ⅓ strips are as long as the 1 strip? Which is longer, a ½ strip or a ⅓ strip?
- ☐ What is the shortest fraction strip in this set? Which strips are longer than this strip?
- ☐ Use the fraction strips to put these common fractions in order, beginning with the largest and ending with the smallest: ⅛, ½, ⅓, ⅙, ¼.
- ☐ Which is longer, the ½ strip or 2 of the ⅓ strips?
- ☐ Which is longer, 2 of the ⅙ strips or 1 of the ¼ strips?
- ☐ Which is shorter, 2 of the ½ strips or 2 of the ⅛ strips?
- ☐ Use strips to put these common fractions in order, beginning with the smallest and ending with the largest: ⅛, ⅔, 2/4, 3/6, 4/12.

Variation: Have children make a fraction-strip book to compare common fractions. Make a unit piece that will be the last page of the book from construction paper. Twelve inches by 2 inches is a good size because 12 is easy to divide into fractional parts. Next make a quarter, two-quarter, and three-quarter piece from construction paper; fold the two-quarter and three-quarter pieces to show their fractional parts. Then make pieces for one-third and two-thirds from another color, and fold the two-thirds-piece. Align the pieces along the left edge with the shortest piece on top and the unit piece at the back. Staple along the left edge. Children can label each fraction and later add additional pages to show sixths and eighths.

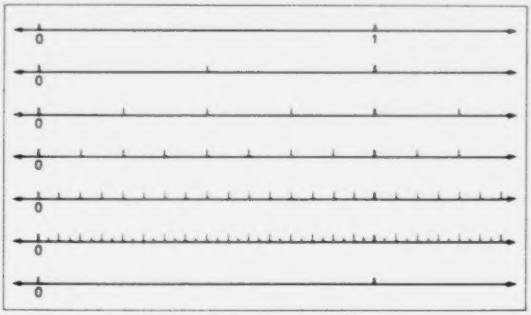


Using the Number Line

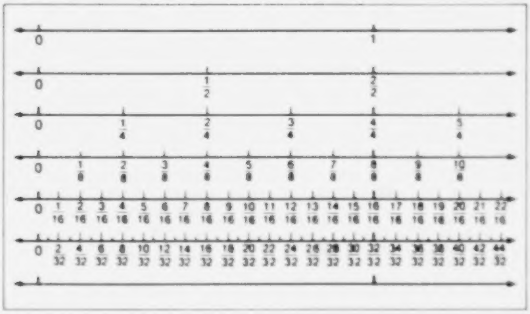
Use the number line extensively with children in grades three through six. There is no better device for comparing fractional numbers and helping children see that there are infinitely many numbers between any pair of fractional numbers.

ACTIVITY 11-5 Number-Line Charts

- Display a chart showing number lines with only the whole numbers named. (Points for fractional numbers are lightly marked to make them easy to locate during the activity.)



- Direct children's attention to the top line to note the unit segment.
- Go to the next line and mark the point midway between the unit segment. Have the children identify and label the points that show ½ and 2/2.
- Go to the next line and mark points on it; have the children identify and label these points: ¼, 2/4, ¾, 4/4, 5/4.
- Continue until the line that shows thirty-seconds has been marked and counted.


0 1
0 1/2 2/2
0 1/4 2/4 3/4 4/4 5/4
0 1/8 2/8 3/8 4/8 5/8 6/8 7/8 8/8 9/8 10/8
0 1/16 2/16 3/16 4/16 5/16 6/16 7/16 8/16 9/16 10/16 11/16 12/16 13/16 14/16 15/16 16/16 17/16 18/16 19/16 20/16 21/16 22/16
0 2/32 4/32 6/32 8/32 10/32 12/32 14/32 16/32 18/32 20/32 22/32 24/32 26/32 28/32 30/32 32/32 34/32 36/32 38/32 40/32 42/32 44/32

- Ask questions like these:
    - ☐ How many of the 1/2 line segments match the length of the 1 segment?
    - ☐ What is the shortest line segment on this chart?
    - ☐ How many 1/16 segments are equivalent to 1/8 segment?
    - ☐ Which is longer, a 1/4 segment or a 1/16 segment?
    - ☐ Which segments are shorter, two 1/8 segments or two 1/4 segments?
    - ☐ What number of eighth segments are equivalent to a half segment? to a fourth segment? to a whole segment?
    - ☐ Use the number line to order these segments from shortest to longest: 1/4, 3/8, 7/16, 19/32, 1/2, 3/4, 30/32, 3/16.

Use the number-line chart to help intermediate-grade children intuitively recognize that subdivision of the line can go on forever and that there are an infinite number of points between any pair of points on the line. Questions like these can be used:

- "If we follow this pattern (the one used in Activity 11-5), what will be the name of the first point on the blank-number line at the bottom of the paper?" (It will be 1/64.)
- "I have run out of space on the chart for more lines and on the lines for more points. Does this mean there are no fractional numbers between 0 and 1/32, 0 and 1/64, 0 and 1/128 . . . ?" (No.)
- "Will the sequence ever end? Why not?" (It is always possible to indicate a point between 2 points on the line.)

Children in the fifth and sixth grades can use the number-line chart to learn how to determine fractional numbers that lie midway between any pair of fractional numbers. They can eventually use the averaging process to find the number midway between a pair of fractional numbers.

ACTIVITY 11-6 Midpoints

- Have children locate the points 2/4 and 3/4 on the fourths line. Have them locate the point midway between them. "What is the simplest name we can give this point?" If children say "five-eighths," ask how they know. If this answer is not given, call attention to the point located at the same position on the eighths line.
- Use questions like these to focus attention on the averaging process:
  "What will 2/4 and 3/4 be called if we express them as eighths?"
  "What will be the sum if we add 4/8 and 6/8?"

"Can you figure out what we can do to the 10/8 to determine the fraction midway between 4/8 and 6/8?" Help children see that the 10 is divided by 2.

- Repeat with other pairs of fractions until the children recognize that by adding a pair of fractional numbers and dividing their sum by 2, they can find a number that lies midway between that pair of fractional numbers.

The computer is useful for exploring common fraction concepts. Programs should be used after children have had activities with concrete models.

In *Fraction 1,*[1] students illustrate common fractions by electronically coloring regions or parts of a set. In *Fraction Fever,*[2] students guide a fraction finder to locate common fractions, collecting points by identifying fractions that are shown in either set or region form. The common fractions vary in complexity and the action is fast paced so that students have to visualize each fraction quickly to find it before time runs out. A strategy for efficiently searching for fractions is required.

Darts and *Number Quest* are programs that use common fraction number lines. Children should complete number-line activities like those in Activity 11-5 and Activity 11-6 before using these programs. In *Darts,*[3] children throw an imaginary dart at balloons on a number line by typing in a numerator and a denominator. If a student chooses a large balloon, the estimates of a common fraction do not need to be as precise as when a small balloon is chosen. *Number Quest*[4] requires knowledge of both common and decimal fractions. Students hunt for a given number by narrowing the range of possible answers by moving toward the center of the line from both the top and the bottom. The calculator is helpful in determining the decimal equivalents of common fractions like 13/16 and 29/32 during the search. A different number quest for whole numbers is on the same disk.

Using Abstract Procedures

Children who have a mature understanding of the procedures for comparing numbers that have already been discussed, who understand the meaning of common fractions, and who can multiply fractional numbers in ways developed later in this chapter, are ready to learn the multiplication process to compare fractions. As you explain the process, give children practice pages containing pairs of common fractions, and help them relate the abstract process to earlier work with models. This is a useful process for dealing with ratios and proportions and is discussed in this connection later.

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[1] Scott, Foresman and Company, 1900 East Lake Ave., Glenview, IL 60025.

[2] Spinnaker Software Corp., One Kendall Square, Cambridge, MA 02142.

[3] *Darts* is found on the Apple Education Classics disk, Apple Computer Inc., 20525 Mariana Ave., Cupertino, CA 95014.

[4] Sunburst Communications Inc., 39 Washington Ave., Pleasantville, NY 10579.

SYSTEMATIC PROCEDURES FOR NAMING EQUIVALENT FRACTIONS

Before children can become proficient in adding and subtracting fractional numbers using common fractions, they must learn to express common fractions so that they have the same denominators. The process for doing this, commonly referred to as changing to a common denominator, must be developed carefully so that children will understand both the necessity for doing it and how to do it meaningfully. Children should learn that when they use 2 equivalent common fractions to rename fractional numbers, the value of each number remains unchanged. Children who have had many experiences with the various learning aids illustrated in this chapter will have little difficulty with numbers such as 1/2, 1/4, 1/8, 1/3, and 1/6. They will "just know" that 1/2 can be renamed as 2/4 or 4/8, 1/4 as 2/8, and 1/3 as 2/6. However, they should learn systematic procedures for dealing with less familiar numbers.

Using Multiples

In some cases the denominator of 1 fraction is a multiple of another. When this is so, as with 2/5 and 3/20, the larger of the 2 denominators is a common denominator. The following thought process can be used:

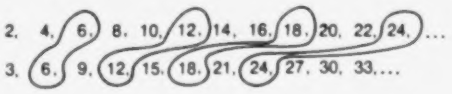
- "Twenty is a multiple of 5. By what number do we multiply 5 to get 20?"
- "To rename 2/5 with a denominator of 20, multiply both the numerator, 2, and the denominator, 5, by 4. Two-fifths is equal to 8/20."

When neither of the denominators is a common denominator, a different procedure must be used. The easiest procedure if denominators are small—for example, 1/2 and 1/3 or 1/4 and 1/6—is to use successive multiples of the denominators until a common denominator has been determined. For 1/2 and 1/3, multiples of 2 and 3 are used.

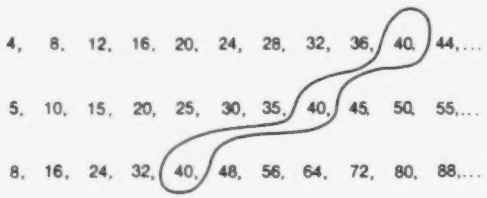
ACTIVITY 11-7

Least Common Multiples

- Use the numbers 2 and 3. Ask, "What are the first several multiples of 2?" "What are the first several multiples of 3?" Ring the common multiples in the 2 lists. Point out that 6 is the smallest (least) number common to both sets of multiples.


2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24, ...
3, 6, 9, 12, 15, 18, 21, 24, 27, 30, 33, ...

- Repeat with other pairs of numbers.
- Show how the process can be used for 3 numbers.


4, 8, 12, 16, 20, 24, 28, 32, 36, 40, 44, ...
5, 10, 15, 20, 25, 30, 35, 40, 45, 50, 55, ...
8, 16, 24, 32, 40, 48, 56, 64, 72, 80, 88, ...

Using Equivalence Classes

Another procedure for finding common denominators is to list the first several numbers in the equivalence class of each number. To add $1/6$ and $3/8$, the following numerals would be listed:

$$\frac{1}{6}, \frac{2}{12}, \frac{3}{18}, \frac{4}{24}, \frac{5}{30}, \frac{6}{36}, \frac{7}{42}, \frac{8}{48}, \ldots$$

$$\frac{3}{8}, \frac{6}{16}, \frac{9}{24}, \frac{12}{32}, \frac{15}{40}, \frac{18}{48}, \frac{21}{56}, \ldots$$

Guide children to see that in the first list there are some fractional numbers that have the same denominators as some in the second listing. These should be indicated: $1/6 = 4/24 = 8/48$ and $3/8 = 9/24 = 18/48$. Children can use both pairs of numbers, $4/24$ and $9/24$ and $8/48$ and $18/48$, to complete the addition of $1/6 + 3/8 = \square$.

$$\begin{array}{ccccc} \frac{1}{6} & \rightarrow & \frac{4}{24} & \rightarrow & \frac{8}{48} \\ +\frac{3}{8} & \rightarrow & +\frac{9}{24} & \rightarrow & +\frac{18}{48} \\ \hline & & \frac{13}{24} & & \frac{26}{48} \end{array}$$

After the sums in the algorithms have been determined, children should note that the answer cannot be simplified when the denominator is 24 (in this example) but can be when it is 48.

3 Using Prime Factorizations

The process of determining common denominators for a pair of unlike fractions can be made systematic by using a factorization process. Children who learn to factor composite numbers can use a factorization process to determine the least common multiple, or least common denominator, when they add or subtract unlike fractions.

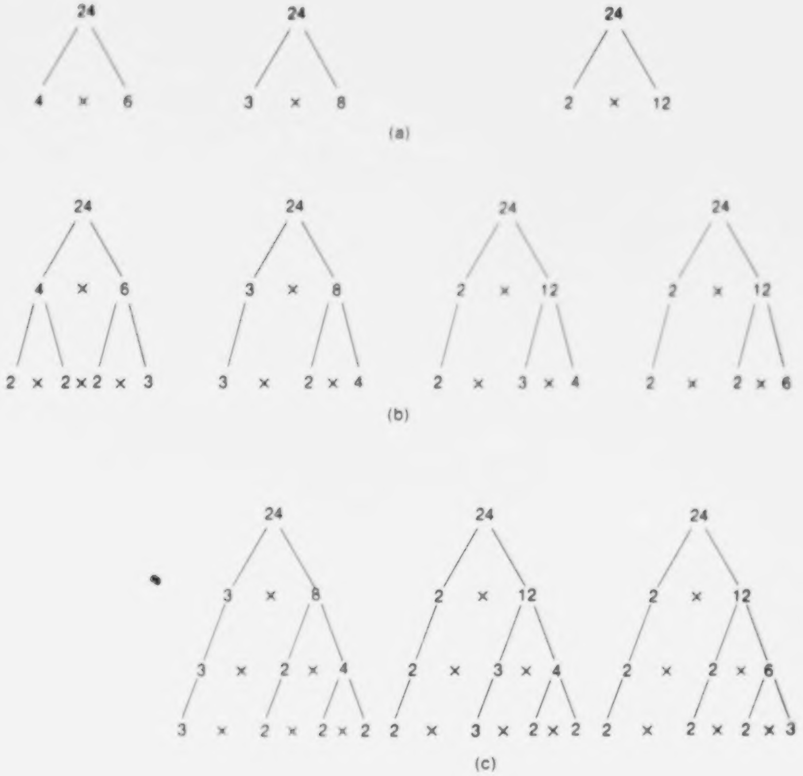
A composite number is said to be factored completely when it is represented as a product expression that consists of 2 or more prime numbers. When the number 18 is factored completely, it is expressed as 2 × 3 × 3; 36 is expressed as 2 × 2 × 3 × 3. Several procedures for factoring composite numbers will be discussed.

a. Use Basic Multiplication Facts Children can begin work with factoring by using small composite numbers, such as 4, 6, 9, 10, 15, and 21. These numbers have only a pair of prime-number factors, so their prime factorization is complete as soon as the 2 factors are named. You can help children use basic facts to factor these numbers: "Which 2 prime numbers are multiplied to give a product of 21?" The basic multiplication fact 3 × 7 = 21 gives the answer.

b. Use Factor Trees Factor trees give a systematic way to factor numbers that are reasonably small but are the product of more than 2 prime numbers. Factor trees are created by expressing numbers in terms of successively smaller factors until all factors are prime numbers. Factor trees for 24 are shown in Figure 11-4. In (a) 3 different ways of beginning are shown; each begins with a different pair of factors. (The number pair 1 and 24 is not used.) In (b) the process is continued with each composite factor expressed by a pair of smaller factors. Note that 12 is factored in 2 ways in the 2 trees on the right, and that the prime factors in any of the trees are represented by a repetition of their numerals. The tree on the left in (b) is completed because in its bottom line, 24 is expressed as the product of prime numbers.

FIGURE 11-4

Different factor trees for the number 24



The completed trees for the other factorizations are shown in (c), where each bottom line expresses 24 as the product of prime numbers. The numerals at the bottom of each completed tree in (c) are 2, 2, 2, and 3.

Other numbers with 2 or more factor trees that can be used for practice are 16, 18, 28, 30, 36, and 40. After children have completed the factor trees for several small composite numbers, they should observe that the same prime numbers are in the bottom line of each number's factorization. For 24 there are 3 twos and 1 three; for 16 there are 4 twos; and so on. Even though children do not have mathematical proof, they can observe that each composite number has only 1 prime factorization, if order of factors is disregarded. This property of numbers is called the *fundamental theorem of arithmetic*.

c. Use an Algorithm Factor trees are useful for numbers small enough for children to easily see at least 1 pair of factors. For larger numbers, an algorithm for factoring numbers is useful. In the example in the

$$\begin{array}{r|r} 2 & 124 \\ 2 & 62 \\ 31 & 31 \\ \hline \end{array}$$

$$\begin{array}{r|r} 3 & 525 \\ 5 & 175 \\ 5 & 35 \\ 7 & 7 \\ \hline \end{array}$$

margin, an algorithm is used to factor 124. First, 124 is divided by 2, and the numeral for the quotient, "62," is written beneath the "124." Sixty-two is also divisible by 2. This division is completed, and the quotient's numeral is written beneath the "62." The number 31 is a prime number, so it is divided by itself. The 3 divisors show the prime factorization of 124 to be 2 × 2 × 31. A second example is shown in the next algorithm in the margin. It shows that the prime factorization of 525 is 3 × 5 × 5 × 7.

In these 2 examples, the division by primes began with the smallest possible prime, which is 2 in the first example and 3 in the second. Each successive division was done with the smallest possible prime, and in both cases a prime number was repeated as a divisor. When the division is done this way, the prime numbers are written in order from smallest to largest. While it is not necessary to take the steps in this particular order, it is recommended for 2 reasons:

1. Children can recognize the divisibility by 2, 3, and 5 more easily than they can by the larger prime numbers.
2. When numbers have been completely factored, the factors are already ordered according to size, and the writing of prime product expressions is simplified. One advantage of writing product expressions in this manner becomes obvious when factoring of numerators and denominators is done to simplify common fractions.

d. Finding Least Common Multiples with Factorization With larger numbers, the process of finding the LCM for 2 or more numbers by listing multiples or finding equivalence classes is difficult and tedious. The use of prime numbers is helpful when large numbers are involved. Begin with simple examples, then use larger numbers.

ACTIVITY 11-8

Factorization Process

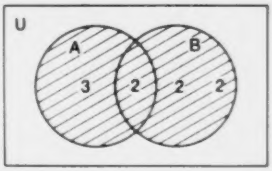
- Begin with an addition-of-fractions situation:

$$1/6 + 1/8 = \square$$

- Determine the prime factorization of each denominator:

$$6 = 2 \times 3 \qquad 8 = 2 \times 2 \times 2$$

- Ask, "What will be the product if we multiply 2 × 3 × 2 × 2 × 2?" (48.) Point out that 2 is common to the factorization of each number, so only 3, rather than 4, twos are needed; the LCM is 2 × 3 × 2 × 2, or 24, not 48.
- A Venn diagram will help make the process clear to children who are familiar with its uses.



- Use a Venn diagram with a pair of larger numbers such as 15 and 12:

$$12 = 2 \times 2 \times 3 \qquad 15 = 3 \times 5$$
$$2 \times 2 \times 3 \times 5 = 60$$

- Children who are familiar with the concept of set union will recognize that the LCM for 2 or more numbers is determined by multiplying the factors that are in the union of factors in the prime factorization of the numbers.

When the process is applied to addition, this sequence of steps is 1 way to proceed. First, determine the prime factorization of each denominator:

$$\frac{4}{15} = \frac{4}{3 \times 5}$$
$$\frac{5}{12} = \frac{5}{2 \times 2 \times 3}$$

Next, form the union of the sets of factors used in the prime factorization of the denominators: $2 \times 2 \times 3 \times 5$. This becomes the denominator that is used in renaming the fractional numbers:

$$\frac{4}{15} = \frac{4 \times 2 \times 2}{2 \times 2 \times 3 \times 5}$$
$$\frac{5}{12} = \frac{5 \times 5}{2 \times 2 \times 3 \times 5}$$

The number by which each numerator is multiplied is found by noting the factors by which the original denominator is multiplied to yield the new denominator: for $4/15$ the factors 2×2 are used, so the numerator is multiplied by 4; for $5/12$ the factor 5 is used, so the numerator is multiplied by 5.

$$\begin{array}{ccccccc} \frac{4}{15} & \rightarrow & \frac{4}{3 \times 5} & \rightarrow & \frac{4 \times 2 \times 2}{2 \times 2 \times 3 \times 5} & \rightarrow & \frac{16}{60} \\ +\underline{\frac{5}{12}} & \rightarrow & +\underline{\frac{5}{2 \times 2 \times 3}} & \rightarrow & +\underline{\frac{5 \times 5}{2 \times 2 \times 3 \times 5}} & \rightarrow & +\underline{\frac{25}{60}} \\ & & & & & & \frac{41}{60} \end{array}$$

RENAMING FRACTIONAL NUMBERS IN SIMPLEST FORM

A fractional number is expressed as a common fraction in its simplest form when the numbers represented by the numerator and denominator have no common factor, or are relatively prime. There are times when the original answers to problems involving fractional numbers are not in their simplest form, and it is desirable to simplify them. Thus after the answer to a subtraction such as 5/8 minus 1/8 has been named as 4/8, it may need to be renamed as 1/2.

Using Number Lines

Number lines are particularly good for helping children learn about expressing common fractions in simplest terms. (You may recognize the process of renaming a fraction in simplest terms as "reducing a fraction to its lowest terms.") During early work with number lines, children use them to compare common fractions and learn that a fraction like 1/2 can be renamed as 2/4, 3/6, 4/8, and so on. At that time, they are changing fractions to higher terms. Now they use the lines to do the reverse; they will simplify fractions, or express them in lowest terms.

ACTIVITY 11-9

Simplifying Fractions

- Use the set of number lines from Activity 11-5.
- Direct the children's attention to 16/32 at the bottom. "What names on other lines name the same point as 16/32?" (8/16, 4/8, 2/4, 1/2.) "Which of these has the smallest denominator?" (1/2.) "Is there a simpler way to name this point?" (No.)
- Discuss the fact that each of the fractions 16/32, 8/16, 4/8, and 2/4 is expressed in simplest form as 1/2.
- Repeat with other points on these lines and with lines showing thirds, sixths, ninths, and twelfths, and fifths and tenths.

Use the fraction strips from Activity 11-4 to simplify common fractions in ways similar to the ways in which number lines are used.

2 Examining Numerator and Denominator

One commonly used procedure is to examine the numerator and denominator to determine the largest number by which both may be divided. Once a common divisor is found, both numerator and denominator can be divided by it. Children with a good understanding of the role of 1 in division will recognize how 1 is used to simplify $^9/_{21}$:

$$\frac{9 \div 3}{21 \div 3} = \frac{9}{21} \div \frac{3}{3} = \frac{9}{21} \div 1 = \frac{3}{7}$$

This procedure is satisfactory as long as the numbers represented by the numerators and denominators are reasonably small and the greatest common divisor can be readily determined, as in $^6/_{12}$ or $^9/_{15}$. It may fail to be useful, however, when the numerators and denominators are larger numbers.

3 Using Prime Factorizations

With larger numbers in the numerator and denominator, a better procedure is to find the prime factorization of each one. An example, $^{24}/_{36}$, will be treated in the same way that children should proceed.

Rewrite each term using its prime factorization:

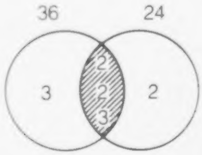
$$\frac{24}{36} = \frac{2 \times 2 \times 2 \times 3}{2 \times 2 \times 3 \times 3}$$

Determine the greatest common factor (GCF) by noting all factors common to both numerator and denominator. (The GCF is made up of the factors in the intersection of the two factorizations. This may be shown with a Venn diagram to help children visualize it. (See Figure 11-5.) Both the numerator and denominator of the fraction should be divided by this common factor to rename it in its simplest form:

$$\frac{24}{36} = \frac{24 \div 12}{36 \div 12} = \frac{2}{3}$$

FIGURE 11-5

Venn diagram showing common prime factors of 36 and 24



OPERATIONS WITH FRACTIONAL NUMBERS EXPRESSED AS COMMON FRACTIONS

By the time children begin to perform the operations of addition, subtraction, multiplication, and division with fractional numbers, they have already developed skill in performing these operations with whole numbers. Armed with this knowledge, teachers frequently introduce these operations with fractional numbers by using a series of rules: "If 2 fractions have the same denominator, their sum can be determined by adding the numerators and placing the sum over the denominator." "If 2 fractions have different denominators, they must be changed to their least common denominator before they can be added." Rules for renaming fractional numbers with a least common denominator are then listed so children can proceed with the addition. However, teaching these operations by rules alone means that many children will have little or no understanding of what they are doing or why they are doing it.

Procedures that help children visualize operations with fractional numbers are extensions of those used with whole numbers. As children learn to add whole numbers, they learn the meaning of the operation in terms of joining sets. Objects, markers, number lines, and other devices are used so that children can discover the meaning of the operation. Children should learn to add fractional numbers in much the same way. Again, physical models help extend children's understanding of addition to include fractional numbers.

ADDITION WITH COMMON FRACTIONS

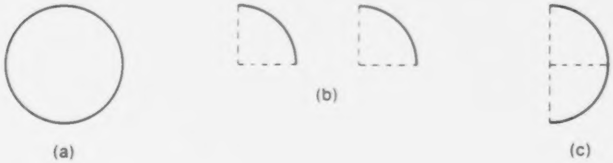
Children's initial experiences with addition should be with familiar situations that are easily represented by physical models.

Adding When Denominators Are the Same

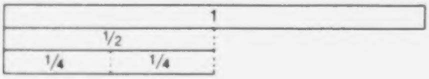
The setting in which children work should be stocked with concrete materials with which children are well acquainted: geometric regions left whole and cut into fractional parts, sets of fraction strips, and number line charts. Present a problem, such as the one in Activity 11-10.

ACTIVITY 11-10 Adding Common Fractions

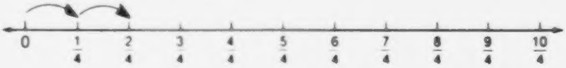
- "Last night Jim practiced his piano lesson for 1/4 of an hour before dinner and 1/4 of an hour after dinner. What part of an hour did he practice altogether?"
- Let each child use concrete materials, if necessary, to determine the answer.
- Discuss each child's way of determining the answer. These are possibilities:
  - "I used fourths of the circle to represent each of the 2 fourth hours. When I put the circles together, I saw that he had practiced for a half hour."



  - "I used 2 pieces that show fourths in our fraction-strip set. They showed me that two-fourths is the same as one-half."



  - "I used the number-line chart. When I took jumps that are each 1/4-unit long, I stopped at 2/4. That is the same as 1/2."



  - Some children may say, "I just know that 1/4 and 1/4 equal 1/2." Some of these children may have used the addition sentence $1/4 + 1/4 = 2/4$ (or 1/2) to determine the answer.
- Present other, similar problem situations to be solved in the same ways.

After children have determined the answers for several problem situations, display the addition sentence for each one and call attention to them: "What have we done in each of these sentences to determine the answer?"

$$\frac{1}{4} + \frac{1}{4} = \frac{2}{4}$$

$$\frac{1}{3} + \frac{1}{3} = \frac{2}{3}$$

$$\frac{1}{2} + \frac{1}{2} = \frac{2}{2}$$

$$\frac{1}{4} + \frac{2}{4} = \frac{3}{4}$$

Guide the children to see that the sum is found by adding the numerators of the fractional numbers. It will be helpful to children to see the sentences rewritten as

$$\frac{1+1}{4} = \frac{2}{4}, \quad \frac{1+1}{3} = \frac{2}{3}, \quad \frac{1+1}{2} = \frac{2}{2}, \quad \text{and} \quad \frac{1+2}{4} = \frac{3}{4}$$

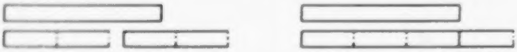
This emphasizes the fact that the numerators are added. If children have difficulty understanding the process, the examples can be further simplified:

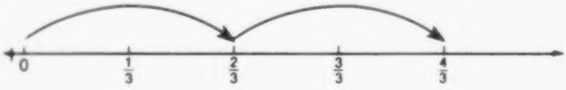
$$\begin{array}{r} \text{1 fourth} \\ \underline{+\text{1 fourth}} \end{array} \qquad \begin{array}{r} \text{1 third} \\ \underline{+\text{1 third}} \end{array} \qquad \begin{array}{r} \text{1 half} \\ \underline{+\text{1 half}} \end{array} \qquad \begin{array}{r} \text{1 fourth} \\ \underline{+\text{2 fourths}} \end{array}$$

Problems with answers greater than 1 can also be examined through activities with concrete materials.

ACTIVITY 11-11 Answers Greater Than One

- "Josie is making a flag and will need ⅔ of a yard of red material and ⅔ of a yard of white material. How much material will she use for the flag?"
- Fraction strips and number line charts are useful here because their properties parallel the linear measure used for determining the amount of material used for the flag.





- Write the addition sentence for this situation:

$$\frac{2}{3} + \frac{2}{3} = \frac{4}{3}$$

- Show that $\frac{4}{3}$ can be renamed as $1\frac{1}{3}$. Emphasize that $\frac{4}{3}$ and $1\frac{1}{3}$ are both names for the same number.
- Repeat with other pairs of fractions.

---

2 Adding When Denominators Are Different

Children should have a good background of skills and understandings before they are introduced to addition of fractional numbers expressed as common fractions with unlike denominators. Included in their background should be

- Understanding of the meaning of addition of whole numbers
- Knowledge of the basic addition facts
- Understanding of common fractions and concrete models that represent them
- Understanding of addition of fractions with like denominators
- Understanding that by using equivalent fractions, fractional numbers can be renamed so they have common denominators

ACTIVITY 11-12

Adding with Unlike Denominators

- "Making some clothes for her sister, Suzy used $\frac{1}{2}$ yard of material for a skirt and $\frac{1}{4}$ yard for a blouse. What part of a yard did she use for the outfit?"
- Some children will perhaps "just know" that the answer is $\frac{3}{4}$ of a yard. Others will probably be uncertain. Use fraction strips as a concrete model of the situation.

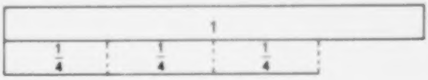
- Write the addition sentence for this situation:

$$\frac{1}{2} + \frac{1}{4} = \square$$

- Discuss what must be done before the addition can be completed. Children who "just know" the answer can perhaps show how the $1/2$ must be renamed as $2/4$ before the addition is completed:

$$2/4 + 1/4 = 3/4$$

- Verify this with the fraction strips by exchanging the half-strip for 2 of the fourth-strips.

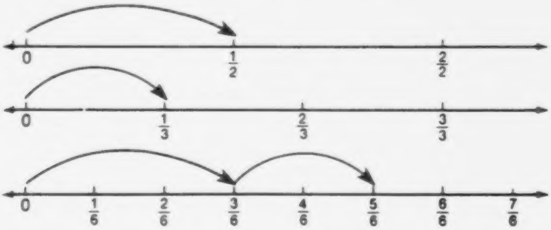


- Repeat with other situations.

The addition sentence $1/2 + 1/3 = \square$ is illustrated with number lines in Figure 11-6. Children will note that it is necessary to rename both fractional numbers so they have common denominators before the addition can take place. To do this, they should study the number lines to locate the line with points that correspond to points on the lines for both thirds and halves. The number lines indicate that a number line showing sixths has points that correspond to points on both the halves and thirds lines. The addition for $1/2 + 1/3 = \square$ is shown when $1/2$ is expressed as $3/6$ and $1/3$ as $2/6$.

FIGURE 11-6

Number lines showing that $1/2 + 1/3 = \square$ is renamed as $3/6 + 2/6 = \square$ to determine the sum $5/6$



Ultimately children who understand least common multiples, or lowest common denominators, and processes for renaming fractions as equivalent fractions will use mature processes for adding fractional numbers. Processes for finding common denominators are discussed in the section of this chapter that deals with systematic procedures for naming equivalent fractions. As children learn these processes, they should also learn to use them when adding fractions with unlike denominators. By completion of elementary school, many children will handle examples such as these without difficulty:

$$\begin{array}{rcl} \frac{3}{4} & \rightarrow & \frac{9}{12} \\ +\frac{5}{6} & \rightarrow & \frac{10}{12} \\ \hline \end{array} \qquad \begin{array}{rcl} \frac{3}{8} & \rightarrow & \frac{9}{24} \\ +\frac{5}{12} & \rightarrow & \frac{10}{24} \\ \hline \end{array} \qquad \begin{array}{rcl} \frac{7}{12} & \rightarrow & \frac{35}{60} \\ +\frac{11}{15} & \rightarrow & \frac{44}{60} \\ \hline \end{array}$$

Adding with Mixed Numerals

Historically the expression *adding mixed numbers* has described addition situations in which both whole and fractional numbers are involved. An example of this addition is

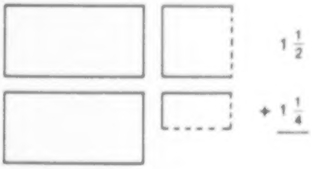
$$\begin{array}{r} 1\,\frac{1}{2} \\ +1\,\frac{1}{4} \\ \hline \end{array}$$

In order to maintain the proper distinction between *number* and *numeral*, the expression *addition with mixed numerals* is a better description of addition of this kind.

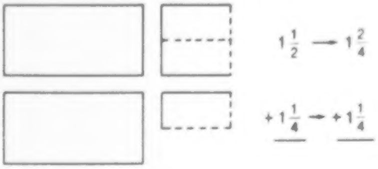
ACTIVITY 11-13

Adding Mixed Numerals

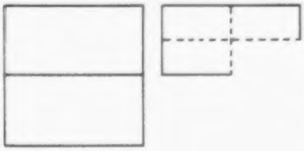
- "Jacques had 1½ pieces of yellow cardboard and 1¼ pieces of blue cardboard. How much cardboard was there when the pieces were put together?"
- Represent the situation with models of rectangles, and write an addition algorithm.



Have the children note that the fractions have unlike denominators; discuss what to do. Show that the ½ piece can be exchanged for two ¼ pieces; rename the ½ as 2⁄4 in the algorithm.


$1\frac{1}{2} \rightarrow 1\frac{2}{4}$
$+1\frac{1}{4} \rightarrow +1\frac{1}{4}$

Combine the pieces in the model; complete the algorithm.



$$\begin{array}{rcr} 1\frac{1}{2} & \rightarrow & 1\frac{2}{4} \\ +\,1\frac{1}{4} & \rightarrow & +\,1\frac{1}{4} \\ \hline & & 2\frac{3}{4} \end{array}$$

Repeat with other mixed numerals.

SUBTRACTION WITH COMMON FRACTIONS

Subtraction of fractional numbers arises from the same kinds of situations that give rise to subtraction with whole numbers. Over a period of time, children should have numerous experiences with each situation by studying problems like these:

1. "If there is ¾ of a pie in a pan and Billy eats a piece that is ¼ of the original pie, how much pie is left?" This is a take-away situation and is represented by circular regions in Figure 11-7.

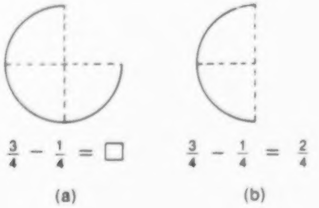

$\frac{3}{4} - \frac{1}{4} = \square$
$\frac{3}{4} - \frac{1}{4} = \frac{2}{4}$
(a)
(b)

FIGURE 11-7

Circular regions used to represent the subtraction sentence ¾ − ¼ = 2/4. (a) ¾ of the unit. (b) 2/4 of the unit.

2. "It is ¾ of a mile from Sue's house to school and ⅞ of a mile from Ted's house to school. How much farther is school from Ted's house than from Sue's house?" This comparison situation is represented with a number line in Figure 11-8.

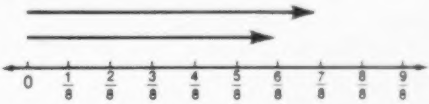


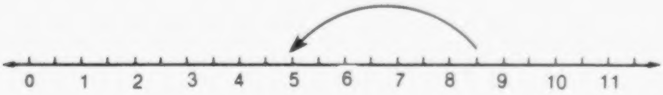
FIGURE 11-8

Number line used to represent the subtraction sentence ⅞ − ¾ = ⅛

3. "Judy is working on a new track layout for her model railroad. She will have a total of $8\frac{1}{2}$ yards of track in the new layout. If she has put down $3\frac{1}{2}$ yards of track, how many more yards of track does she have left to put down? This situation is represented by a number line in Figure 11-9.

FIGURE 11-9

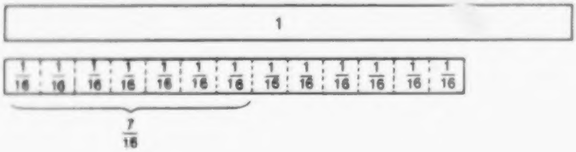
Number line used to represent the subtraction sentence $8\frac{1}{2} - 3\frac{1}{2} = 5$



4. "Carmen has a mixture of 2 chemicals that weighs $13/16$ of a pound. She has $7/16$ of a pound of sulfur in the mixture. How much of the mixture is made up of the other chemical?" This situation is illustrated by fraction-strip pieces in Figure 11-10.

FIGURE 11-10

Fraction strips used to illustrate the subtraction sentence $13/16 - 7/16 = 6/16$



Children already know that subtraction and addition are inverse operations, so they learn to add and subtract fractional numbers simultaneously. You should use the same care to introduce subtraction through situations that are relevant to them. Children's first subtraction should be with fractions having like denominators; problems 1 and 4 illustrate suitable situations and appropriate learning aids. Subtraction with fractions having unlike denominators, as in problem 2, should be introduced along with addition with unlike denominators. In this way the need for renaming fractions so as to result in common denominators will be clear. The renaming process is the same for either operation.

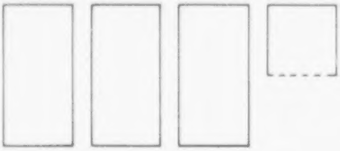
Subtraction with mixed numerals is troublesome for some children, especially when the common fraction part of the minuend is smaller than the common fraction part of the subtrahend. A child may rename the subtrahend as in the margin. Here the child has used the same procedure for renaming the mixed numeral as used earlier for regrouping when subtracting whole numbers, renaming it as $11/4$, rather than $5/4$. To avoid this type of error, begin instruction for this kind of subtraction with situations such as the one that follows.

$$\begin{array}{r} 6\frac{1}{4} \\ -3\frac{3}{4} \\ \hline \end{array} \qquad \begin{array}{r} 5\frac{11}{4} \\ -3\frac{3}{4} \\ \hline \end{array}$$

ACTIVITY 11-14

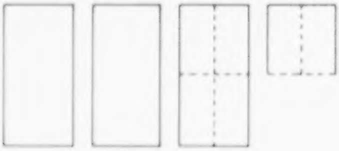
Cake Sale

- "Roberto was in charge of the cake sale at his school's bazaar. At 8:00 he counted 3½ cakes on the table. By 8:30 he had sold 1¾ of these cakes. How many cakes were left at 8:30?"
- Show models of the cakes at 8:00.



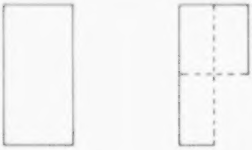
$$\begin{array}{r} 3\frac{1}{2} \\ -1\frac{3}{4} \\ \hline \end{array}$$

- Discuss what must be done before 1 and ¾ cakes can be removed; cut 1 cake into fourths and combine it with the half cake, which has also been cut into fourths.



$$\begin{array}{r} 3\frac{1}{2} \\ -1\frac{3}{4} \\ \hline \end{array} \rightarrow \begin{array}{r} 3\frac{2}{4} \\ -1\frac{3}{4} \\ \hline \end{array} \rightarrow \begin{array}{r} 2\frac{6}{4} \\ -1\frac{3}{4} \\ \hline \end{array}$$

- Remove 1¾ from the model, leaving 1¾; complete the algorithm.



$$\begin{array}{r} 3\frac{1}{2} \\ -1\frac{3}{4} \\ \hline \\ \end{array} \rightarrow \begin{array}{r} 3\frac{2}{4} \\ -1\frac{3}{4} \\ \hline \\ \end{array} \rightarrow \begin{array}{r} 2\frac{6}{4} \\ -1\frac{3}{4} \\ \hline 1\frac{3}{4} \end{array}$$

- Relate each step in the algorithm with each step in the model.
- Repeat with similar situations and models.

MULTIPLICATION WITH COMMON FRACTIONS

The operations of multiplication and division of fractional numbers are deceptively easy for teachers to teach and children to perform, but their meanings are elusive. Children can be taught rules for performing these 2 operations, and as long as they remember the rules, they can multiply or divide pairs of fractional numbers with ease. However, if children learn to perform these operations using only rules, they probably will understand very little of the meaning behind them. If you adopt the philosophy of this book, you will use learning aids, not simply the rules, as you explore with children the meanings of the operations.

Multiplication of Fractional Numbers by Whole Numbers

Multiplication of fractional numbers by whole numbers is usually used for beginning work, because the examples are easily related to the repeated-addition interpretation of multiplication. When a group of children is ready, use a realistic situation.

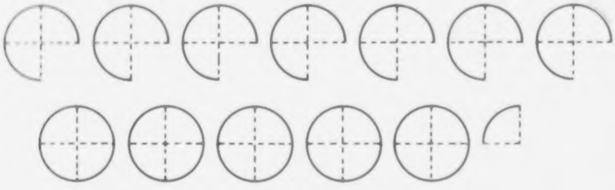
ACTIVITY 11-15

Multiplying a Fraction by a Whole Number

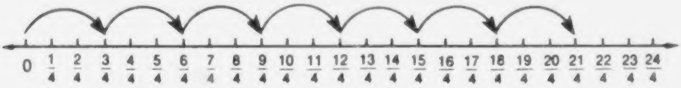
- "Sarah practices the piano ¾ of an hour each day. What is the total amount of time she practices in a week?"
- Discuss the meaning of this problem, then have the children give a multiplication sentence:

$$7 \times \tfrac{3}{4} = \square$$

- Use models to represent the situation. Circular regions, each showing ¾, are good because they can be related to ¾ of an hour on the clock. First, show ¾ seven times. Then count the total number of fourths (there are 21). Show the 21 fourths as 5 whole circles and ¼ of another circle.



A number line is another easily handled model.



- Complete the multiplication sentence. Rename the answer as a mixed numeral:

$$7 \times \tfrac{3}{4} = \tfrac{21}{4}$$
$$\tfrac{21}{4} = 5\tfrac{1}{4}$$

- Call children's attention to the fact that the $5\frac{1}{4}$ represents hours.
- Repeat with similar problems.

---

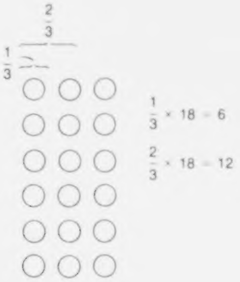
2 Multiplication of a Whole Number by a Fractional Number

Many practical situations give rise to the need for finding a fractional part of a whole number. When one cube of butter weighs ¼ of a pound, the weight can be expressed as ¼ of 16 ounces, or 4 ounces. One-third of a foot is the same as ⅓ of 12 inches, or 4 inches; ½ of a dollar is ½ of 100 pennies, or 50 cents. Frequent encounters with these and similar applications enhance children's appreciation of the need for knowing how to perform such multiplication.

ACTIVITY 11-16

Multiplying a Whole Number by a Fraction

- Display a 3-by-6 array of disks. Have the children recall the multiplication sentence for this array: 6 × 3 = 18.
- "I want to set ⅓ of this array off to 1 side. How many disks should I set aside?"



- Discuss the fact that to determine ⅓ of the set it is necessary to separate the disks into 3 equal-sized groups; one of the groups is ⅓ of the 18 disks. "How many disks are in ⅓ of the set?" (6.)

- "How many disks are in ⅔ of the 18 disks?" (12.)
- Repeat with other sets that can be separated into equal-sized groups without remainder.

Once children have used arrays many times, help them learn that the multiplication of a pair of numbers, such as ⅔ and 18, can be completed by either of these procedures:

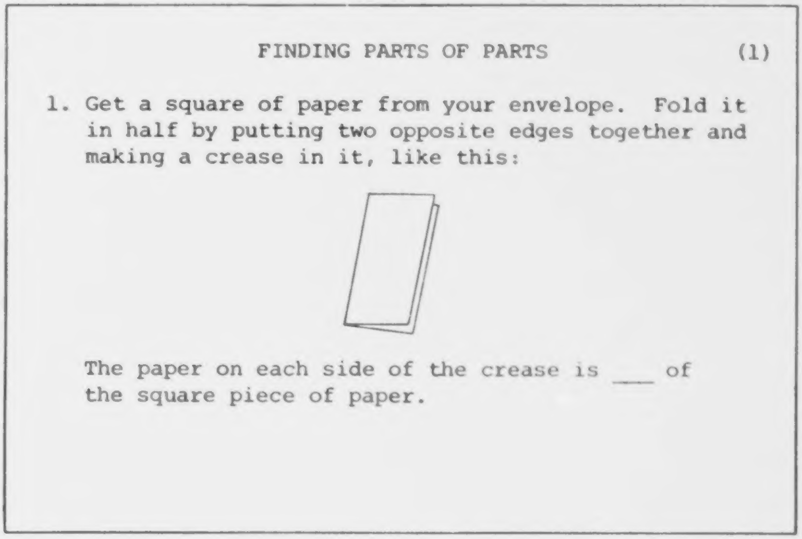
$$\frac{2 \times 18}{3} = \frac{36}{3} = 12 \qquad \text{or} \qquad \frac{18}{3} \times 2 = 6 \times 2 = 12$$

3 Multiplication of Fractional Numbers by Fractional Numbers

A paper-folding learning center can be used to give children an interesting approach to multiplication of 1 fractional number by another. Problem cards give directions, and paper cut in the shapes of squares, circles, and rectangles provides the materials for investigations. A manila envelope containing a set of shapes and a response sheet for each person makes it easy for a child to begin work. Sample cards are shown in Figure 11-11.

FIGURE 11-11

Sample paper-folding problem cards



FINDING PARTS OF PARTS (1)

1. Get a square of paper from your envelope. Fold it in half by putting two opposite edges together and making a crease in it, like this:

The paper on each side of the crease is ___ of the square piece of paper.

FIGURE 11-11
(continued)

(2)

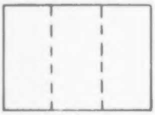
2. Fold the paper one more time so that it looks like this:

Now, the paper is folded so there are ___ sections, or parts. Each part is ___ of the square I began with.

3. After your first fold, each section was ___ of the square. When you made the second fold, each section was ___ of the square. This shows that 1/2 of 1/2 is ___ of the whole piece of paper.

(3)

4. Take out the sheet of paper that looks like this:



Fold on the dotted lines. Each section of the paper is ___ of the whole rectangle.

5. Fold the paper so that you fold the thirds in half. What is the size of each section now? This shows that 1/2 of 1/3 is ___ of the whole piece of paper.

Prepare other cards for activities with thirds, fourths, sixths, and eighths of fractions and for nonunit fractions like $\frac{2}{3}$ and $2\frac{3}{4}$ of units.

When children have completed the center activities, discuss their work with them to help them summarize their findings. Overhead transparencies are useful for a review of the paper-folding activities. The transparencies for the sentence $\frac{2}{3} \times \frac{3}{4} = \frac{6}{12}$ are illustrated in Figure 11-12. In (a) a unit square is shown. In (b) $\frac{3}{4}$ of the unit is represented by shading on an overlay. In (c)

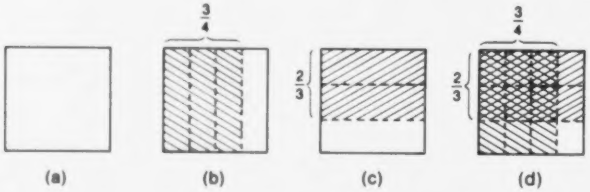


FIGURE 11-12

The multiplication sentence $\frac{2}{3} \times \frac{3}{4} = \frac{6}{12}$ illustrated with a series of overhead transparencies

another overlay represents $\frac{2}{3}$ of the unit. Finally, $\frac{2}{3}$ times $\frac{3}{4}$ is illustrated when both overlays are placed over the unit region at the same time. The answer, $\frac{6}{12}$, is represented by the six doubly shaded sections.

As children work with their various models to gain an understanding of multiplication of fractional numbers, their attention should be directed so they will see the relationship between the fractions' numerators and the product and the denominators and the product. Eventually they should generalize that to multiply fractional numbers, they multiply the numbers represented by numerators and the numbers represented by denominators to get the product. Thus the answer for the sentence $\frac{3}{4} \times \frac{7}{8} = \square$ is found by multiplying 3 times 7 to name the numerator of the product and 4 times 8 to name its denominator.

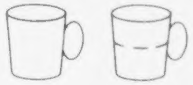
Multiplication Involving Mixed Numerals

An extension of multiplication with pairs of whole numbers and pairs of fractional numbers is made when multiplication is done with what are commonly referred to as *mixed numbers*. The phrase *multiplication with mixed numerals* more properly explains this type of multiplication, because it maintains a distinction between number and numeral. An example of this multiplication is $2\frac{1}{2} \times 1\frac{1}{2} = \square$.

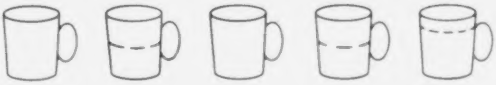
ACTIVITY 11-17

Multiplying with Mixed Numerals

- "A cookie recipe calls for $1\frac{1}{2}$ cups of flour. If you want $2\frac{1}{2}$ times as many cookies as 1 recipe makes, how many cups of flour will you need?"
- Discuss the situation and portray the $1\frac{1}{2}$ cups of flour.



- "You want $2\frac{1}{2}$ times as many cookies as you will get if you make the recipe once. Can you figure out how much flour you will need?" Help the children see that they will need one more $1\frac{1}{2}$ cupfuls, plus $\frac{1}{2}$ that much again, or $\frac{3}{4}$ of another cup.



- "Altogether, how much flour will you use?" ($3\frac{3}{4}$ cups.)
- Write the multiplication sentence for this situation:

$$2\frac{1}{2} \times 1\frac{1}{2} = \square$$

Rename each mixed numeral as an improper fraction:

$$\frac{5}{2} \times \frac{3}{2} = \square$$

Complete the multiplication:

$$\frac{5}{2} \times \frac{3}{2} = \frac{15}{4} = 3\frac{3}{4}$$

- Use other examples for the same recipe:
  - The recipe calls for $1\frac{1}{4}$ teaspoons of baking soda. How much is needed for $2\frac{1}{2}$ recipes?
  - The recipe calls for $\frac{3}{4}$ teaspoon vanilla. How much is needed for $2\frac{1}{2}$ recipes?

---

5 Cancellation

(a) $\frac{2}{3} \times \frac{9}{10} = \square$

(b) $\frac{2 \times 9}{3 \times 10} = \square$

(c) $\frac{2 \times (3 \times 3)}{3 \times (2 \times 5)} = \square$

(d) $\frac{2 \times 3 \times 3}{2 \times 3 \times 5} = \square$

(e) $\frac{2}{2} \times \frac{3}{3} \times \frac{3}{5} = \square$

(f) $1 \times 1 \times \frac{3}{5} = \square$

Cancellation is a process of simplifying the common fractions used in multiplication before the work is done rather than after. (Cancellation also works in certain division situations.)

The rationale for the process is given by the examples in the margin. The original multiplication sentence is shown in (a). Rewrite the sentence as in (b). Next, factor the composite numbers as in (c). Application of the commutative property permits rewriting the sentence as in (d). In (e) the sentence is written in a way that emphasizes fractions that are names for 1. Finally, 2 of the fractions are rewritten as ones, and the multiplication is completed. Children with a good understanding of one's role in multiplication should have little difficulty in understanding this rationale.

Children will complete the process without following all of the steps. You will help them recognize that they can cancel when they find a numerator and denominator that are alike, as in the first of the following examples, or when there are numerators and denominators that have common factors as in the other two examples:

$$\frac{4}{5} \times \frac{3}{4} = \square$$

$$\frac{3}{4} \times \frac{8}{9} = \square$$

$$\frac{5}{12} \times \frac{3}{10} = \square$$

DIVISION WITH COMMON FRACTIONS

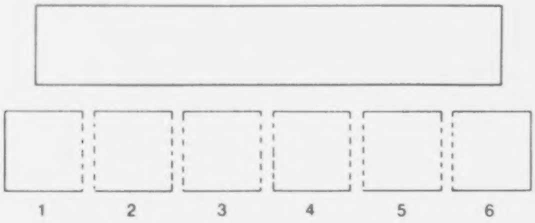
Situations That Illustrate Division with Common Fractions

Help children relate division with fractional numbers to division with whole numbers and extend their understanding of both partitive and measurement situations to include fractional numbers. As before, early experiences should include work with concrete models to aid in visualizing problem situations and the role of division in their solution. Introductory lessons should begin with a review of a situation used with whole numbers. The examples need not involve large numbers.

ACTIVITY 11-18 Measurement Division with Fractions

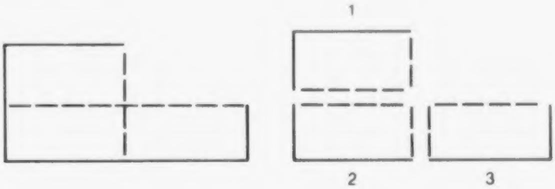
Put 15 markers on a magnetic board. "If we put these markers into groups that each have 3 markers, how many groups will there be?" Review the idea that this measurement problem can be solved by repeated subtraction.

Show a rectangular region and this situation: "I have a piece of lumber 3 feet long. If I cut it into pieces that are each ½ foot long, how many pieces will I have?"



Help the children see that this is also a measurement situation. Once the board has been cut, the ½-foot-long pieces can be removed one at a time. There are 6 pieces.

Show a model of ¾ of a cake and present this situation: "I have ¾ of a cake. When I cut it into pieces that are each ¼ of the cake, how many pieces will there be?"



This is also a measurement situation. Count the ¼-sized pieces as they are removed 1 at a time. There are 3 pieces.

Repeat with other situations that yield whole-number answers.

A similar series of activities should be used to introduce partitive situations. Use examples like these:

> "There is ¾ of a pie in Jack's refrigerator. If he cuts it into 3 equal-sized pieces, what part of the whole pie will each piece be?"
>
> "A half gallon of ice cream is divided into 6 equal-sized portions. What part of a gallon is each portion?"

Division Algorithms

Two algorithms can be used for dividing with common fractions. Most commonly taught in the elementary school is the invert-and-multiply algorithm, which is based on the idea that dividing 1 number by another is the same as multiplying the number by the reciprocal of the divisor. Dividing 6 by 3 is the same as multiplying 6 by ⅓; 4 ÷ ½ equals 4 × 2. The algorithm is shown for these 2 division situations:

$$6 \div 3 = 6 \times \frac{1}{3} = 2$$

$$4 \div \frac{1}{2} = 4 \times \frac{2}{1} = 4 \times 2 = 8$$

The other algorithm requires that both fractions have the same denominator. Division is done by dividing the numerator of the dividend by the numerator of the divisor. To divide $\frac{1}{2}$ by $\frac{1}{4}$, rename $\frac{1}{2}$ as $\frac{2}{4}$ and then divide 2 by 1. This series of steps shows this algorithm.

$$\frac{1}{2} \div \frac{1}{4} = \frac{2}{4} \div \frac{1}{4} = \frac{2 \div 1}{4 \div 4} = \frac{2}{2} = 2$$

Because the invert-and-multiply process is the most widely used algorithm, it is the only one explained in this book for presentation to all children. You should avoid presenting this algorithm without developing adequate understanding of how and why it works.

a. Reciprocals Before children are introduced to the invert-and-multiply process, they should have an understanding of reciprocals. They can be introduced to this concept through an activity like this:

ACTIVITY 11-19 Finding Reciprocals

- Show the children this set of multiplication sentences:

$$\frac{3}{4} \times \square = \frac{12}{12}$$

$$\frac{6}{7} \times \square = \frac{42}{42}$$

$$\frac{2}{3} \times \square = \frac{6}{6}$$

- Ask these questions:
  - ☐ "By what number must we multiply $\frac{3}{4}$ to obtain the product $\frac{12}{12}$?" ($\frac{4}{3}$.)
  - ☐ "By what do we multiply $\frac{6}{7}$ to obtain a product of $\frac{42}{42}$?" ($\frac{7}{6}$)
  - ☐ "By what do we multiply $\frac{2}{3}$ to obtain the product $\frac{6}{6}$?" ($\frac{3}{2}$.)
- After the second factor for each sentence has been determined, ask these questions:
  - ☐ "What do you notice about each product in these sentences?" (They are all names for 1.)
  - ☐ "What do you notice about the 2 factors in each example?" (The numerator and denominator of the second are the reverse of the first.)

- "What is the product of each of the following sentences?"

$$\frac{4}{15} \times \frac{15}{4} = \square$$

$$\frac{9}{7} \times \frac{7}{9} = \square$$

$$2 \times \frac{1}{2} = \square$$

- After children know how to determine a reciprocal, help them compose a definition of it. For example, "The reciprocal of a number is the number by which the original number is multiplied to yield a product of 1."
- Variation: Have children compute answers to these sentences on a calculator:

$$(3 \div 4) \times (4 \div 3) = \square$$

$$(4 \div 15) \times (15 \div 4) = \square$$

---

b. Invert and Multiply When children can both describe measurement and partitive situations that involve fractions and understand reciprocals, they are ready to comprehend the invert-and-multiply algorithm for dividing with common fractions. The algorithms that follow are based on the problems in Activity 11-18.

ACTIVITY 11-20 Using Reciprocals

- Begin with the sentence for the lumber situation in Activity 11-18:

$$3 \div \tfrac{1}{2} = 6$$

- Ask questions like these:
  - "What is the reciprocal of $\frac{1}{2}$?" ($\frac{2}{1}$, or 2.) "What happens if I multiply 3 by the reciprocal of $\frac{1}{2}$?" (The product is 6.)
- Show the sentence for the cake situation in Activity 11-18:

$$\tfrac{3}{4} \div \tfrac{1}{4} = 3$$

- Use these questions:
  - "What is the reciprocal of $\frac{1}{4}$?" ($\frac{4}{1}$, or 4.) "What happens if I multiply $\frac{3}{4}$ by $\frac{4}{1}$ and simplify the answer?" (The product is 3.) Repeat with the other examples you used earlier.
- Help the children see that answers to these division situations are obtained by inverting the divisor and then multiplying. Show the sequence of steps for each situation like this:

$$3 \div \frac{1}{2} = 3 \times \frac{2}{1} = 3 \times 2 = 6$$

$$\frac{3}{4} \div \frac{1}{4} = \frac{3}{4} \times \frac{4}{1} = \frac{3}{4} \times 4 = \frac{12}{4} = 3$$

Guide the children through as many examples as are necessary for them to relate the invert-and-multiply algorithm to both measurement and partitive situations. When its use is clear, help them state a generalization such as: "To divide fractional numbers, multiply the dividend by the reciprocal of the divisor."

RATIOS

One of the uses of common fractions discussed at the beginning of this chapter is that of indicating ratios. When common fractions are used to indicate ratios, they are interpreted differently than when they represent parts of a whole or group. For example, the numerator of a common fraction that represents a ratio may tell the number of elements in 1 of 2 sets, whereas the denominator tells the number of elements in the other set. Ratios cannot be added, subtracted, multiplied, and divided like fractional numbers either. Consequently you must use care when common fractions are introduced as ways of representing ratios so that their meaning is clear.

Rate Pairs and Tables

You do not need to delay introduction of ratios until children have mastered all the other uses of common fractions. A good way to make the introduction is through work with rate pairs. A simple example shows how this can be done:

ACTIVITY 11-21

Rate Pairs

- "I bought 2 pencils for five cents. Will one of you use these play coins and these pencils to show us 2 pencils and five cents?"
- "How much money will I need in order to buy 4 pencils at the same price?"
- Through questions and use of common materials, you can help children understand the many-to-many correspondences involved in this situation.
- Show a simple table and discuss how it can be completed.

Number of Pencils	2	4	6				
Cost of Pencils	5¢	10¢	15¢				

- Repeat with other common situations and materials, such as: 3 lollipops for nineteen cents, 4 gumballs for thirty-five cents, and 5 mini-boxes of raisins for fifty-nine cents.

Proportions

Proportional relationships are frequently used in problem solving. Consider this situation:

Mary and Sue share a newspaper route. Mary delivers papers 1 day out of every 3; Sue delivers the papers 2 days out of every 3. For every day that Mary delivers papers, Sue delivers them on 2 days; this is a ratio of 1 to 2. If Sue has delivered papers for a total of 30 days, on how many days has Mary delivered them?

This proportion can be used to solve the problem:

$$\frac{1}{2} = n/30$$

Help children see that this problem can be solved by determining the number by which 2 is multiplied to get a product of 30, and then multiplying 1 by the same number. Also help children see that the process is the same as they have used to rename fractional numbers to find common denominators.

The missing term in any proportion can be determined by cross multiplying and then dividing. To find the value of n in the expression $6/7 = n/42$, multiply 6×42 and then divide by 7. When 42 is multiplied by 6, the product is 252; $252 \div 7 = 36$, so $6/7 = 36/42$. Naturally, more than one example needs to be used so that cross multiplication is clear to children.

Much of the elementary-school mathematics program dealing with common fractions needs to emphasize concrete models and practical situations because children do not develop a mature understanding of proportionality until the age of twelve or thirteen. Therefore you must not attempt to take the children too deeply into a study of proportionality.

The use of proportions in problem solving is discussed further in the next chapter, in which their use in solving percent problems is considered.

SPECIAL PROBLEMS WITH COMMON FRACTIONS

Some children frequently experience difficulties as they learn about common fractions. Children with visual discrimination or figure–ground weaknesses often find it hard to distinguish a part from the whole when a geometric region represents a part-of-a-whole situation and a set of disks represents a part-of-a-set situation. The parts of a common fraction numeral are equally baffling for many of these children. Children with memory deficits cannot always retain information acquired during work with concrete models long enough to transfer it to a symbolic form as they learn about numerals for common fractions. Integrative problems lead to difficulties with computational processes with common fractions.

Specially prepared materials and the careful development of their meanings will help lessen the difficulties these children experience. Each algorithm must be introduced through situations and with models that are clear to each child. An early and careful analysis of children's written computational procedures will enable you to deal with problems as they arise and prevent faulty procedures from becoming habitual.

Models like those used throughout this chapter are less confusing for many children than the ones used in many textbooks. In this book, the region or strip that represents a whole is outlined with a solid line, and the parts into which the region or strip is separated are shown with broken lines. This model is easier for children to visualize than when solid lines alone are used. When these models are combined with the use of color cues, as suggested by Nancy Bley and Carol Thornton,[5] children can see the sequence to follow as they interpret the model and write the fraction's numeral. The model and numerals in Figure 11-13 might be used. The original geometric figure and the parts into which it is separated are outlined

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[5] Nancy S. Bley and Carol A. Thornton, *Teaching Mathematics to the Learning Disabled* (Rockville, Md.: Aspen Systems Corporation, 1981), p. 312.

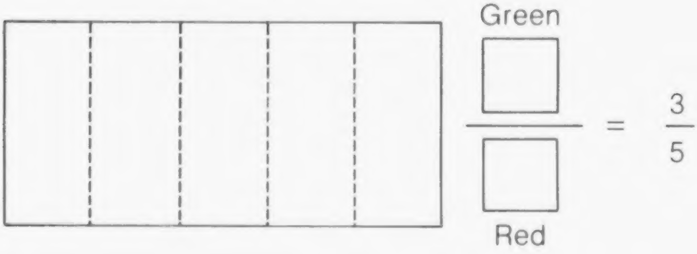


FIGURE 11-13

Geometric model and colored boxes for learning to interpret a common fraction

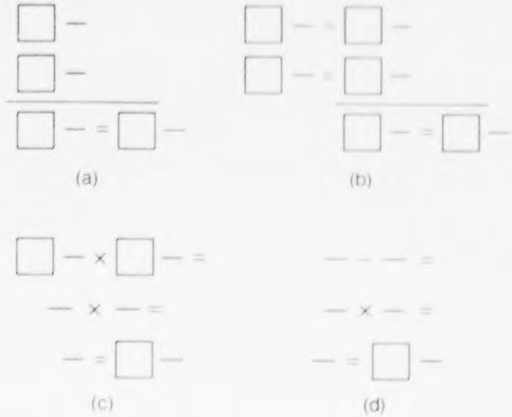
in red, and the parts being considered are colored green. The box in the numerator position is green, and the denominator box is red. The child counts the green parts and writes a "3" in the green box. Next, all of the parts are counted, and a "5" is written in the red box. Finally the "3⁄5" is written.

Numerals that represent fractional numbers as common fractions are confusing for many children. Whole-number numerals are written horizontally, beginning at the left and ending at the right, but common fractions are written with 1 numeral above another. Mixed numerals are especially confusing for some. Sandpaper or felt numerals and flat boxes with wet sand, as suggested in Chapter 6 for children who have difficulty writing the numerals for whole numbers, are useful now. As a child traces a mixed numeral, the fraction it represents should be named by the child. Then the numeral should be copied with chalk on a chalkboard or with pencil on paper. An adult or another student can work with a child to direct the work and determine the correctness of responses.

An additional problem encountered by some children is that of organizing numerals in algorithms correctly. Bley and Thornton recommend that a preformatted page be used.[6] Figure 11-14 (a) shows a model for adding when there are mixed numerals with like denominators. The whole number for each addend is placed in the box; the numerator for the fraction goes above the line and the denominator below the line. A page of 8 or 10 models is given to a child who cannot align numerals in an algorithm. Each example from an assignment is copied in 1 of the models before any answers are determined. The second model (b) is for adding mixed numerals with unlike denominators. These same models can be used for subtraction also. The models in (c) and (d) are for multiplying mixed numerals and dividing fractions, respectively.

FIGURE 11-14

Patterns for computing with fractional numbers



Summary

Common fractions have several meanings, depending upon the situation in which they are used. Children learn that common fractions are used to express parts of a unit or set of objects that has been subdivided into equal-sized parts or subsets, to express ratios, to indicate division, and to express abstract fractional numbers. Children need experience with concrete and semiconcrete models that represent all of the meanings. Geometric regions, fraction strips, markers, and number lines are used during investigative activities designed to make the common-fraction representation of fractional numbers meaningful. These same materials, along with programs for computers and certain abstract procedures, help children learn to compare fractional numbers, to rename them as common fractions in their simplest form, and to rename 2 or more fractional numbers so they are expressed by common fractions having the same denominator. Factorization of numbers is an abstract process that can be used to simplify common fractions and to find lowest common denominators.

Operations with fractional numbers expressed as common fractions require the same careful development that is made for operations with whole numbers. Even though children are older when operations with fractional numbers are developed, it is still necessary that the processes be developed through carefully sequenced activities with appropriate learning aids rather than by rules alone. Real-life problems can illustrate situations that give rise to addition, subtraction, multiplication, and division with fractional numbers. Geometric regions, fraction strips, markers, and number lines should accompany the real-life problems to give further meaning to the operations. References to similarities that exist between the opera-

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[6] Bley and Thornton, p. 305.

tions with whole numbers and those with fractional numbers should be stressed whenever it is appropriate.

There are procedures for developing children's understanding of how cancellation is done when multiplying fractional numbers. Uses of common fractions to express ratios should be introduced through simple examples at first, with more complex examples introduced as children mature.

Some children experience difficulty with common fractions because models and numerals are not always understood by them. Models that use color cues, sandpaper and felt numerals, wet-sand boxes for writing numerals, and algorithm patterns are useful aids for these children.

Study Questions and Activities

1. Find new activities and games in the articles listed in the For Further Reading section to add to your teaching file. Look for articles that describe games and concrete materials in each of the following areas: understanding and naming fractional regions, understanding and naming fractional parts of a set, naming or renaming equivalent fractions, comparing unequal fractions, and understanding addition, subtraction, multiplication, and division of fractions.
2. Read the article by Usiskin or by Usiskin and Bell (see For Further Reading section) and summarize arguments for common fractions in the elementary-school mathematics curriculum. Compare their ideas to Joseph Payne's article on curricular issues. How do you feel about the need for common fractions? State your position and tell how this position agrees or differs with the articles mentioned.
3. Word problems are frequently used in this chapter to present situations leading to operations with common fractions. Develop your own skill by making up a word problem for a real-life situation that corresponds with each of the following number sentences. Also draw a picture or use materials to model how each operation would be done.

   (a) $\frac{3}{8} + \frac{3}{8} = \frac{6}{8}$ (b) $1\frac{1}{2} + \frac{1}{4} = 1\frac{3}{4}$
   (c) $\frac{7}{8} - \frac{3}{8} = \frac{4}{8}$ (d) $3 - 1\frac{1}{2} = 1\frac{1}{2}$
   (e) $\frac{2}{3} \times 18 = 12$ (f) $2 \times \frac{3}{8} = \frac{6}{8}$
   (g) $6 \div \frac{1}{2} = 12$ (h) $\frac{1}{3} \div 2 = \frac{1}{6}$

4. One way to assess fifth- or sixth-grade students' understanding of fraction sentences is by having them make up word problems and use concrete materials or pictures to demonstrate a process for solving each problem. Use number sentences from Study Question 3 as you interview several children. Have each child make up a word problem and then demonstrate its solution with concrete materials or drawings. If you find a child who cannot successfully model his or her own prob-

lems, you can do the modeling and ask the child to explain what you are doing. Children who cannot create situations that correspond to the sentences show a lack of understanding of common fractions and their uses. As a teacher, you would need to consider reteaching common fraction concepts with concrete materials and familiar problem situations to develop their understanding of processes.

5. Examine a current mathematics textbook series to determine the placement of common fraction concepts throughout the series. The information will be contained in the scope-and-sequence chart in the teacher's manual. At what grade level is the concept of ratio introduced? What are some of the situations used to help children understand ratios? When is the concept of proportion introduced? What examples are used to illustrate this concept?
6. If you are familiar with Logo, you can use a computer to try the procedures developed by Tipps (see For Further Reading section) that allow students to play with common fractions. Both region and number-line formats are provided. If you are working with children, see how they respond to these procedures.

For Further Reading

Beedle, Ruby B. "Dot Method of Renaming Fractions," *Arithmetic Teacher*, XXXIII, No. 2 (October 1985), pp. 44–45. Demonstrates transitional method for renaming fractions.

Brown, Christopher N. "Fractions on Grid Paper," *Arithmetic Teacher*, XXVI, No. 5 (January 1979), pp. 8–10. Describes and illustrates grid-paper activities to develop fundamental concepts of common fractions and visual recognition of parts and wholes and to give students experiences with equivalent fractions.

Carlisle, Earnest. "Fractions and Popsicle Sticks," *Arithmetic Teacher*, XXVII, No. 6 (February 1980), pp. 50–51. Natural and colored popsicle sticks serve as models for multiplying and dividing with common fractions. Examples are described and illustrated.

Chiosi, Lou. "Fractions Revisited," *Arithmetic Teacher*, XXXI, No. 8 (April 1984), pp. 46–47. Diagrams provide multiple ways of showing fractions.

Ellerbuck, Lawrence W., and Joseph W. Payne. "A Teaching Sequence from Initial Fraction Concepts through the Addition of Unlike Fractions," in *Developing Computational Skills*, 1978 Yearbook, National Council of Teachers of Mathematics (Reston, Va.: The Council), pp. 129–147. The sequence includes initial work with common fractions using objects and diagrams and leading to oral names and fraction symbols. The concepts are extended to include work with equivalent fractions, comparing, and renaming. Addition of like and unlike fractions is described briefly.

Hollis, L. Y. "Teaching Rational Numbers—Primary Grades," *Arithmetic Teacher*, XXXI, No. 6 (February 1984), pp. 36–39. Activities and games for introducing fractions.

Jacobson, Marilyn Hall. "Teaching Rational Numbers—Intermediate Grades," *Arithmetic Teacher*, XXXI, No. 6 (February 1984), pp. 40–42. Enhance understanding of rational numbers and operations with activities and models.

Nelson, Rebecca S., and Donald R. Whitaker. "Another Use for Geobards," *Arithmetic Teacher*, XXX, No. 8 (April 1983), pp. 34–37. Geo-islands provide many fractional activities.

Payne, Joseph N. "Curricular Issues: Teaching Rational Numbers," *Arithmetic Teacher*, XXXI, No. 6 (February 1984), pp. 14–17. Raises several questions about the teaching of fractions and decimals and makes recommendations about instruction.

Prevost, Fernand J. "Teaching Rational Numbers—Junior High School," *Arithmetic Teacher*, XXXI, No. 6 (February 1984), pp. 43–46. Operations with fractions are easier when children have a variety of ways of representing them.

Silvia, Evelyn M. "A Look at Division with Fractions," *Arithmetic Teacher*, XXX, No. 5 (January 1983), pp. 38–41. Graphic display of division with fractions increases understanding of the operation.

Skypek, Dora Helen B. "Special Characteristics of Rational Numbers," *Arithmetic Teacher*, XXXI, No. 6 (February 1984), pp. 10–12. Rational numbers can be represented in many forms and have several different meanings.

Souviney, Randall J. "Seeing Through Fractions," *Learning*, V, No. 7 (March 1977), pp. 66–67. Regions cut from colored transparent acetate serve as models for developing the meaning of common fractions. Later they are used to give meaning to addition, subtraction, and multiplication with common fractions.

Swart, William L. "Fractions vs. Decimals—the Wrong Issue," *Arithmetic Teacher*, XXIX, No. 2 (October 1981) pp. 17–18. Swart argues that greater emphasis needs to be placed on concepts of fractions and less on computation with them. He uses an interesting problem to present his case.

Sweetland, Robert D. "Understanding Multiplication of Fractions," *Arithmetic Teacher*, XXXII, No. 1 (September 1984), pp. 48–52. Suggests activities with Cuisenaire rods that promote understanding of fractions and the multiplication algorithm.

Thompson, Charles. "Teaching Division of Fractions with Understanding," *Arithmetic Teacher*, XXVII, No. 5 (January 1979), pp. 24–27. Illustrates and explains a set of materials and a sequence of steps for teaching the common-denominator method of dividing with common fractions.

Tipps, Steve. "Funatic Fractions," *The Logo Exchange*, V, No. 2 (October 1986), pp. 6–8. Describes Logo procedures to create a Fraction Maker for demonstrating fractional regions and addition of fractions.

———. "Fraction Line Frolics," *The Logo Exchange*, V, No. 3 (November 1986), pp. 4–8. Describes Logo procedures for creating a number line and comparing fractional values on it. Addition and subtraction of fractions are also included.

Trafton, Paul R., and Judith S. Zawojewski. "Teaching Rational Number Division: A Special Problem," *Arithmetic Teacher*, XXXI, No. 6 (February 1984), pp. 20–22. Proposes an instructional sequence with activities moving from common fractions to decimal fraction division.

Usiskin, Zalman P. "The Future of Fractions," *Arithmetic Teacher,* XXVII, No. 5 (January 1979), pp. 18–20. Usiskin refutes the argument that common fractions will become obsolete by exploring uses of fractions that will persist regardless of the import of either the calculator or the metric system.

Usiskin, Zalman, and Max S. Bell. "Ten Often Ignored Applications of Rational Number Concepts," *Arithmetic Teacher,* XXXI, No. 6 (February 1984), pp. 48–50. Facts about rational numbers help teachers and students understand and use them.

Van de Walle, John, and Charles S. Thompson. "Fractions with Counters," *Arithmetic Teacher,* XXVIII, No. 2 (October 1980), pp. 6–11. Sets of teacher-made unit holders and markers form the basis for activities that develop understanding of common fractions, equivalent fractions, and addition and subtraction of fractions.

———. "Fractions with Fraction Strips," *Arithmetic Teacher,* XXXII, No. 4 (December 1984), pp. 4–9. Shows how fraction strips make fractions more interesting and understandable.

Wiebe, James H. "Discovering Fractions on a 'Fraction Table'," *Arithmetic Teacher,* XXXIII, No. 4 (December 1985), pp. 49–51. Number lines taped onto a table provide many activities with fractions.

Chapter

12

Fractional Numbers—Decimal Fractions and Percent

In Chapter 12, you will read about

1. materials and procedures for introducing decimal fractions to children
2. how a calculator can help children understand decimal fractions
3. materials and procedures for teaching children the four operations with decimal fractions
4. using estimation to learn how to place decimal points in products and quotients involving decimal fractions
5. computer and calculator activities dealing with decimal fractions
6. the meaning and uses of percent
7. materials and procedures for introducing percent
8. problem solving with percent

Decimal fractions have assumed a prominent position in each person's repertoire of mathematical knowledge and skills. For example, knowledge and understanding of what a calculator is and how it works requires knowledge of the Hindu–Arabic numeration system extended to include decimal fractions. Likewise, knowledge of decimal fractions is essential for understanding the metric system. Knowledge of percent is also important for today's increasing use of credit and automatic-teller-machine cards and consumer loans, for an understanding of many daily news stories, as well as for success in many occupational endeavors. This chapter discusses materials and procedures for helping children learn about decimal fractions and percent and their applications in realistic situations.

INTRODUCING DECIMAL FRACTIONS

Decimal fractions are one of 3 ways to represent fractional numbers. Studying them should be related to what has been learned about common fractions and the Hindu–Arabic numeration system. Models for decimal fractions should be similar to those used to learn about common fractions so that relationships between the 2 ways of representing fractional numbers can be emphasized. In many instances, middle-grade children can learn about common and decimal fractions simultaneously, using the same models. This approach has 2 benefits. First, children learn that common fractions and decimal fractions both represent the same kind of numbers rather than viewing them as being unrelated, as is often the case when they are studied separately. Second, it saves time, because many of the same concrete and semiconcrete materials can be used simultaneously to develop an understanding of both types of fractions.

You should also make clear that decimal fractions are an extension of the already familiar base-ten numeration system.

Once it is clear that children are ready to learn about decimal fractions, familiar concrete aids should be used for developing their understanding. Cuisenaire rods or base-ten blocks are useful if they are available. Children who are accustomed to these materials will know that any particular piece can be named as a unit. Therefore they can use a tens flat from a Cuisenaire or base-ten set as a unit. Ten rods cover a flat, so each rod is a tenth of the flat; while 100 small unit squares cover a flat, and each one is a hundredth of a flat.

Squared paper with centimeter squares can be cut to make an activities kit. A kit, which each child can make, consists of squares that are 10 units along each side, at least 10 strips that are 10 units long, and 100 small unit squares. If squared paper is not available, you can draw a kit's pattern on duplicating masters and make copies of the parts for each child.

Tenths

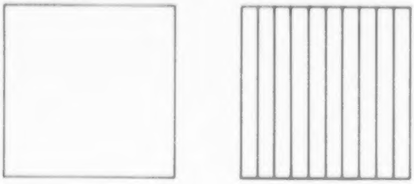
When a group of third- or fourth-grade children is ready to work with decimal fractions, give each child a kit of materials containing either flats and rods from a set of Cuisenaire or base-ten materials, or squares and strips cut from tagboard.

ACTIVITY 12-1

Tenths

Tell the children that for this lesson the flat from the Cuisenaire or base-ten set or the squares of tagboard represent units, or ones.

Have each child lay 10 rods or strips of tagboard on top of a unit piece.



Have the children verify that there are 10 rods or strips on the unit piece.

Ask for a volunteer to give a number name for 1 rod or strip. ($\frac{1}{10}$.)

Ask for volunteers to give names for 2 of the smaller pieces, 3 pieces, 4 pieces, and so on until 9 pieces have been named.

Ask a volunteer to come to the board to write a numeral to name 1 of the small pieces. Accept either $\frac{1}{10}$ or 0.1 (.1). If $\frac{1}{10}$ is written, ask if anyone can write the same number another way. Likewise, if 0.1 is written first, ask for the alternate notation. Point out that $\frac{1}{10}$ and 0.1 are 2 names (numerals) for the same number.

Do the same for the other fractional parts up to 0.9.

Tell the children that 1.0 is the numeral to use when a unit has been cut into 10 parts and all 10 parts are being considered.

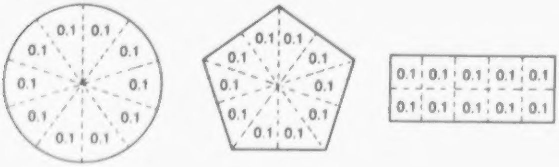
In Activity 12-1, the notations 0.1 and 1.0 were used. You should be aware that it is common practice to put a zero in front of the decimal point when a decimal fraction is written. When used this way, the zero helps make it clear that a decimal fraction is being considered. When a numeral for a decimal fraction is written without the zero, it is easy to overlook the decimal point and misread the numeral. The zero after the decimal point in 1.0 indicates that a unit has been separated into 10 parts and that all 10 parts are being considered. So the zero has significance in this situation and should not be omitted.

Children should engage in activities with other materials until they have a well-developed understanding of tenths and decimal notation for tenths. Present lessons and activities with materials like the ones that follow.

Geometric shapes. Children use shapes other than squares that are separated into 10 parts. Help them identify and name each part as $\frac{1}{10}$ and to write decimal notation for 1 part, 2 parts, and so on (Figure 12-1).

FIGURE 12-1

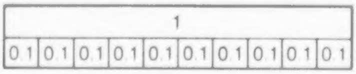
Geometric shapes separated into 10 congruent parts to illustrate tenths



Fraction strips. Use a unit strip and a strip separated into 10 congruent parts. Review the names and notation for each part and collections of parts up to 1.0 (Figure 12-2).

FIGURE 12-2

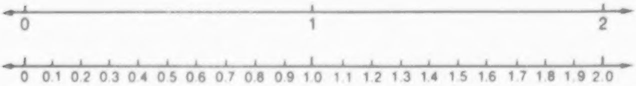
Fraction strips showing the decimal-fraction representation of tenths



Number line. Display a number line that is 20 inches long between 0 and 1, and between 1 and 2 (Figure 12-3). Show a congruent number line beneath the first one. Mark 10 congruent spaces between 0 and 1 and between 1 and 2. Have children identify the name for each point between the whole numbers. (This model provides a basis for a discussion of the mixed numerals between 1 and 2.)

FIGURE 12-3

Number lines used to illustrate units and tenths



Hundredths

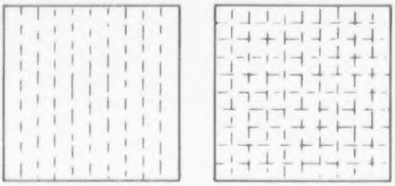
Develop children's understanding of the decimal-fraction representation of hundredths through extension of activities with tenths. The same materials used for learning about tenths can be used, except that materials to represent hundredths must be added to the kits.

ACTIVITY 12-2

Hundredths

Display a unit region. Separate it into 10 congruent parts; review the notation for decimal tenths.

Separate each tenth into 10 congruent parts so there are 100 parts.



Discuss the fact that there are now 100 congruent parts. Introduce the notation for one hundredth: "0.01."

Have the children count a few of the parts; display this sequence of numerals as they count:

0.01, 0.02, 0.03, ...

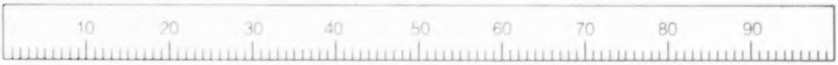
Name decimal fractions for children to represent with kit materials: 0.15, 0.36, 0.98, and so on.

Show representations of decimal fractions with models; have the children name each fraction.

A meter stick separated into decimeters and centimeters is useful for helping children understand tenths and hundredths (Figure 12-4). Children can examine a stick to note that there are 10 decimeters and 100 centimeters. Have a child or the class in unison repeat the names "one-tenth, two-tenths, three-tenths, . . . ten-tenths" as each child moves a finger along a stick. Help them note that there are 10 hundredths (centimeters) in each decimeter.

FIGURE 12-4

A meter stick shows tenths and hundredths of a meter



Other Decimal Fractions

Learning aids like those used for tenths and hundredths are impractical to use for representing fractional numbers smaller than hundredths. However, children can see unit regions that have been separated into 1,000 parts to extend their understanding of notation to include thousandths. From there it is not unreasonable for children to *imagine* that such a unit has been cut into 1,000, or 10,000, of even 100,000 parts as they deal with the smaller decimals. In the same way that a number line can be used to help children realize that there is no smallest common fraction, you can help them realize that theoretically a unit region can be cut into smaller and smaller pieces to represent smaller and smaller decimal fractions and that there is no smallest decimal fraction.

EXTENDING PLACE VALUE TO INCLUDE FRACTIONAL NUMBERS

Kits of structured or squared-paper materials and other learning aids provide a basis for developing the concept of decimal fractions as numerals for fractional numbers. By using these materials, children learn that 0.3, 0.13, and 1.3 are equivalent to the common fractions $\frac{3}{10}$, $\frac{13}{100}$, and $1\frac{3}{10}$, respectively, and that each pair names the same number. These same activities and the ones that follow should also be used to emphasize the place of decimal fractions in the Hindu–Arabic numeration system. Children must see how decimal fractions provide place positions to the right of the ones place to accommodate numbers smaller than one.

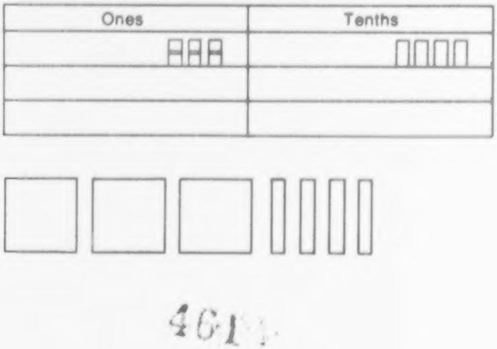
Using Place-value Pocket Charts

A place-value pocket chart, a type-1 proportional aid, is useful for helping children understand how decimal fractions are an extension of the already familiar Hindu–Arabic numeration system.

ACTIVITY 12-3

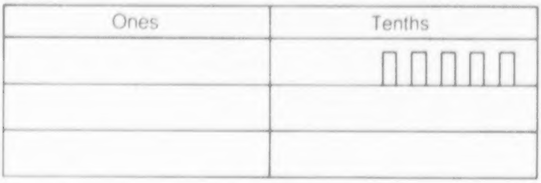
Decimal Fractions

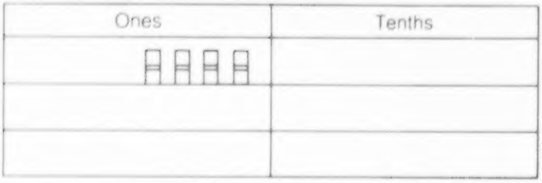
Display both a pocket chart and a set of kit materials. A set of 3 unit and 4 tenth pieces are shown here. The pocket chart contains 3 bundles of tenths pieces and 4 single tenths pieces.



Point out that the 3 bundles of tenths represent 3 units, while the single pieces represent 4 tenths. Write the numeral 3.4 on the chalkboard and discuss its meaning.

Repeat with other decimal fractions, including some with no tenths and some with no units. In the first pocket chart that follows, there are no units; the correct numeral for this is 0.5. In the second chart, there are no tenths; the numeral is 4.0.





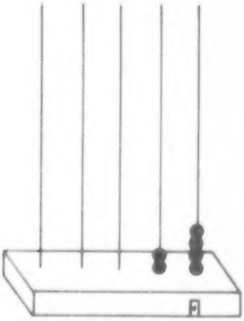
Dictate or give written numerals for numbers like 1.5, 4.7, 5.0, and 0.3 for children to represent with kit materials and the pocket chart.

Using an Abacus

When an abacus is used to represent decimal fractions, the ones position must be someplace other than at the right end of the device. In Figure 12-5, the ones place is the second rod from the right; the right-hand rod accom-

FIGURE 12-5

The number 2.4 represented on an open-ended abacus



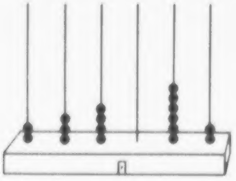


FIGURE 12-6

The number 234.062 represented on an open-ended abacus

modates tenths. A decimal point is indicated by a dot on a small square of paper between the ones rod and the tenths rod. The fractional number 2.4 is shown on this abacus. As children mature, they need to see how more complex fractional numbers fit into the Hindu–Arabic system. The abacus in Figure 12-6 shows the number 234.062.

Using a Calculator

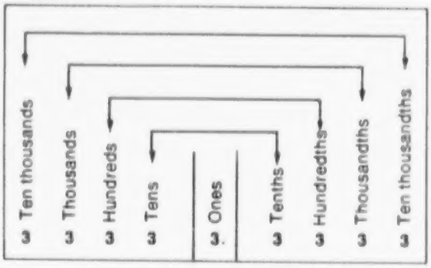
One of the first activities for helping children learn to use a calculator teaches them how to enter numbers into the machine. They learn that to enter a number like 342, they enter a 3, a 4, and a 2, in that order. During early study of decimal fractions, they should learn how to enter numbers represented by these numerals into a calculator. Activities devoted to helping them learn to do this also strengthen children's understanding of decimal notation. They should learn that to enter the number shown on the abacus in Figure 12-5, they enter a 2, the decimal point, and a 4, in that order. For the number represented in Figure 12-6, they enter the 234, then the decimal point, and finally the 0 tenths, 6 hundredths, and 2 thousandths, in that order. Once children enter a number, help them to read it. The number 2.4 is read "two point four" and "two and four tenths." The number 234.82 is read "two hundred thirty-four point eighty-two," "two three four point eight two," and "two hundred thirty-four and eighty-two hundredths."

Children can use Wipe Out (Activity 7-5) with decimal fractions as well as whole numbers. Now, instead of wiping out tens or hundreds or thousands, children wipe out tenths, hundredths, or thousandths on their calculators. An ability to use a calculator correctly to wipe out the hundredths in this number—32.67—and thousandths from a number like 4.283 is a good indication that a child understands both the machine and decimal fractions.

Pocket-chart, abacus, and calculator work should be followed by an activity that helps children summarize ideas about place value. Emphasize that it is the ones place, not the decimal point, that is the point of symmetry for place value in the Hindu–Arabic numeration system. Also, each place-value position has a value that is 10 times that of the position to its right and $^1/_{10}$ that of the position to its left, regardless of where it is in relation to the ones place. A chart like the one shown in Figure 12-7 clearly shows that the ones place is central to the system; it stands alone. Other positions such as

FIGURE 12-7

Place-value positions of a number indicated on a chart



tens and tenths are positioned to the left and right, respectively, an equal number of places from the ones place.

As children work with whole numbers, they learn to use expanded notation to interpret the meaning of different numbers. First they learn that a whole number such as 6382 can be expressed as 6000 + 300 + 80 + 2. By the end of elementary school they should be able to express these ways:

$$(6 \times 1000) + (3 \times 100) + (8 \times 10) + 2$$
$$(6 \times 10 \times 10 \times 10) + (3 \times 10 \times 10) + (8 \times 10) + 2$$
$$(6 \times 10^3) + (3 \times 10^2) + (8 \times 10) + 2$$

By using expanded notation with decimal fractions, children will further their understanding of these numerals.

Initial exercises such as the following should involve filling in the blanks:

2.36 = ____ones____tenths____hundredths
0.46 = ____ones____tenths____hundredths
2.482 = ____ones____tenths____hundredths____thousandths

Later exercises should stress more abstract symbolism:

$4.68 = 4 + (\underline{\qquad} \times 0.1) + (\underline{\qquad} \times 0.01)$
$2.43 = 2 + (4 \times \underline{\qquad}) + (3 \times \underline{\qquad})$
$2.006 = 2 + (\underline{\qquad} \times 0.1) + (\underline{\qquad} \times 0.01) + (\underline{\qquad} \times 0.001)$

Finally, by the end of grade six, some children will be ready to use exponential notation to represent decimal fractions. Those with a mature understanding of negative numbers can use the final form shown here:

$$343.68 = (3 \times 10^2) + (4 \times 10) + 3 + (6 \times 1/10) + (8 \times 1/10^2)$$
$$343.68 = (3 \times 10^2) + (4 \times 10^1) + (3 \times 10^0) + (6 \times 10^{-1}) + (8 \times 10^{-2})$$

ADDING AND SUBTRACTING WITH DECIMAL FRACTIONS

Children who have developed a good understanding of fractional numbers and their decimal-fraction representations will have little difficulty adding and subtracting them. Fractional numbers expressed with decimals are added and subtracted just like whole numbers expressed with Hindu–Arabic numerals.

Adding with Decimal Fractions

Realistic situations with which children can easily relate should be used to introduce addition involving decimal fractions. When you use a word problem to introduce the operation, you help children recognize the types of situations that result in addition with decimal fractions and to realize the importance of knowing how to do such addition. Activity 12-4 introduces addition with tenths. A situation involving addition with hundredths follows the activity.

ACTIVITY 12-4

Adding with Decimal Fractions

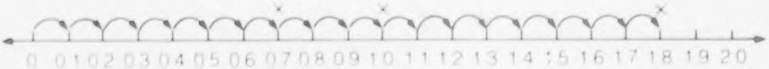
"Judy uses her bicycle for errands. She keeps track of distance with a metric odometer (which measures kilometers) attached to her bicycle. One day she recorded these distances: 0.7, 0.3, and 0.8. How many kilometers did she ride that day?"

Have children write a mathematical sentence for this situation:

$$0.7 + 0.3 + 0.8 = \square$$

$$\begin{array}{r} 0.7 \\ 0.3 \\ +0.8 \\ \hline \end{array}$$

If necessary have children use materials to represent the decimal fractions. A number line is illustrated here.

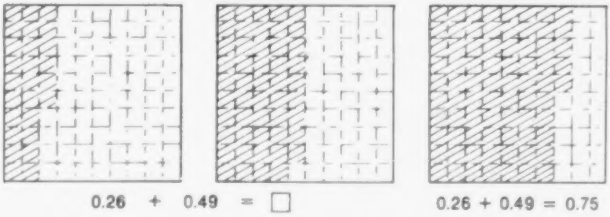


$$\begin{array}{r} 0.7 \\ 0.3 \\ +0.8 \\ \hline 1.8 \end{array}$$

Write an algorithm, as in the margin.
Add the numbers shown in the algorithm.
Repeat with similar problem situations.

FIGURE 12-8

The addition sentence 0.26 + 0.49 = 0.75, illustrated with parts of a unit region



Addition with hundredths can be illustrated in similar ways. In Figure 12-8, the addition sentence 0.26 + 0.49 = □ is represented with parts of a unit region. First, 26 of the 100 congruent parts are shaded to represent the first addend; then 49 parts of another square are shaded to represent the second addend. When the 2 sets of shaded parts are considered together, they indicate the sum, 0.75.

Addition with decimal fractions can be represented with place-value pocket charts and an abacus, also. These devices are used with decimal fractions in the same way as with whole numbers. Procedures with whole numbers are described in Chapter 9.

Subtracting with Decimal Fractions

Two subtraction algorithms for whole numbers are discussed in Chapter 9. Either of the processes can be used with decimal numbers. Whichever algorithm your children have mastered is the one they should use now. It is still important to use problem situations so children will learn about real-world applications of the new subtraction.

ACTIVITY 12-5

Subtracting with Decimal Fractions

"A piece of jewelry wire is 63.2 centimeters long. A piece that is 12.6 centimeters long is cut from it. How long is the remaining piece?" Have the children give the subtraction sentence for this situation:

$$63.2 - 12.6 = \square$$

Write an algorithm:

$$\begin{array}{r} 63.2 \\ -12.6 \\ \hline \end{array}$$

Discuss the fact that it is not possible to do decomposition subtraction without regrouping from the ones to the tenths place.

Complete the algorithm:

$$\begin{array}{r} 6\overset{2}{\cancel{3}}.{}^{1}2 \\ -1\,2.\,6 \\ \hline 5\,0.\,6 \end{array}$$

Repeat with similar problems.

MULTIPLYING WITH DECIMAL FRACTIONS

Because fractional numbers expressed as decimal fractions are multiplied using the same algorithm form as whole numbers, you should guard against having this process become mechanical for children. Introductory work with such multiplication should be presented within a familiar context relevant to other aspects of the children's lives.

Multiplication Situations

The first type of multiplication is illustrated by a problem-solving situation in which a meter stick is a helpful model. For the second type, a set of overlays for an overhead projector serve as a model; while for the third, a strip of calculator tape is used.

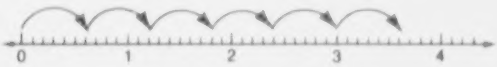
ACTIVITY 12-6 Multiplying a Decimal Fraction by a Whole Number

"Jack is repairing his railroad track layout. He has found that he will need 6 pieces of wire, each 0.6 of a meter long. How much wire will he need altogether?"

Point out that this is a repeated-addition situation; the multiplication sentence is:

$$6 \times 0.6 = \square$$

Draw lines that are each 0.6 of a meter end to end with a meter stick to illustrate the multiplication.



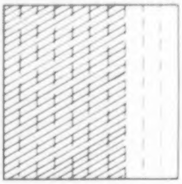
Discuss the meaning of the answer: 3.6 meters.

Repeat with pieces of wire of different lengths and quantities.

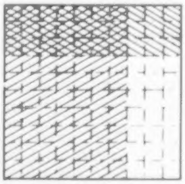
ACTIVITY 12-7 Multiplying a Decimal Fraction by a Decimal Fraction

"Art foil comes in sheets that are 10 centimeters square. Sarah has 0.7 of a full sheet. If she uses 0.3 of this piece for an art project, how much of an original sheet will she use?"

Show a square projected on the screen by an overhead projector. Shade 0.7 of the square to indicate the size of Sarah's piece of foil.



Use horizontal lines to separate each of the tenths into 10 parts. Shade the top 3 tenths to show the 0.3 that Sarah will use.



The answer is represented by the 21 parts that were shaded twice. Each of the 21 parts represents one of the 100 parts into which the entire unit was separated, so the product is 0.21.

Write the complete multiplication sentence:

$$0.3 \times 0.7 = 0.21$$

Repeat with other examples of the same sort.

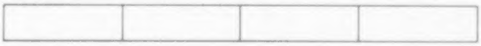
ACTIVITY 12-8 Multiplying a Whole Number by a Decimal Fraction

"A piece of ribbon that is 4 meters long is cut so that 0.2 of it is removed. What is 0.2 of 4 meters?"

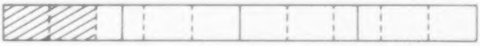
Write the multiplication sentence:

$$0.2 \times 4 = \square$$

Illustrate with a 4-meter-long piece of calculator tape, marked to show the 4 meters.



Mark the piece of tape into 10 congruent parts to show tenths of the whole piece. Shade 2 of the tenths to indicate the 0.2 that is being cut off.



Measure the shaded parts in relation to 1 meter. The model shows that 0.8 of 1 meter is removed when 0.2 of 4 meters is removed.

Write the complete sentence:

$$0.2 \times 4 = 0.8$$

Repeat with similar examples.

Placement of the Decimal Point in a Product

After visualizing the meanings of the 3 types of multiplication situations with the help of models, children will multiply by hundredths and thousandths. They need to learn a useful way for determining where to place the decimal point in each product. Eventually they should be guided to generalize that the number of decimal places in a product equals the sum of the decimal places to the right of the decimal point in each of the factors.

Before they use this generalization, they should use estimation to help them understand it. To estimate a product for the sentence $0.9 \times 2.1 = \square$, they should think: "0.9 is nearly 1, while 2.1 is about 2; 1 × 2 is 2." The only reasonable placement of the decimal point in the actual product is between the 1 and 8 because 1.89 is about 2. For this sentence—$11.2 \times 23.4 = \square$—the thinking is "11.2 is about 10 and 23.4 is about 25; 10 times 25 is about 250." The product is correct when the decimal point is between the 2 and 0 because 262.08 is about 250. Children who need practice in making estimates to determine the placement of decimal points can complete exercises containing examples like these, where they are to underline the correct product for each one.

$2.4 \times 6.8 =$	16.32	1.632	163.2	1632
$10.6 \times 3.68 =$	3900.8	390.08	39.008	3.9008
$9 \times 2.98 =$	2682	268.2	2.682	26.82

After children have completed exercises such as these, ask questions to help them discover the generalization about determining the number of decimal places in a product. "How many decimal places are there following the one in the first factor of this sentence?" "How many are there in the second factor?" "What is the sum of the number of places to the right of the ones in both factors?" "How many decimal places are there to the right of the ones place in the product?" "Look at the next sentence. What do you notice about the number of decimal places to the right of the ones place in both of its factors and in its product?" "Now look at the remainder of the sentences. What do you notice about them?" "Can anyone tell how to determine where to put the decimal point in a product when the factors are decimal fractions?"

DIVIDING WITH DECIMAL FRACTIONS

Division involving decimal fractions determines either the size of each group when a set is divided into a given number of groups, or the number of groups when a set is subdivided into groups of a given size, just as it does in division of whole numbers. The first of these is a partitive situation; the second is a measurement situation. The algorithm forms used for division of whole numbers are also used for division of fractional numbers expressed as decimal fractions. The concepts and algorithms are the same for division of both kinds of numbers. The new element is the treatment of the decimal point in the quotient. A logical explanation, not a memorized rule, is needed to help children learn to place it properly.

Division Situations

This activity introduces a partitive situation:

ACTIVITY 12-9

Partitive Division

"A rope is 16.4 meters long. If Jane cuts it into 4 pieces of equal length, how long will each piece be?"

Calculator tape is a good model to use. Cut a piece that is 16.4 meters long. Discuss the problem to determine what is to be done, and write a division sentence and an algorithm:

$$16.4 \div 4 = \square \qquad 4\overline{)16.4}$$

- "Will each piece be at least 1 meter long?" (Yes.) "Two meters long?" (Yes.) "Three meters long?" (Yes.) "Four meters long?" (Yes.) "Five meters long?" (No.)
- Show the division of 16 by 4 in the algorithm:

$$\begin{array}{r} 4 \\ 4\overline{)16.4} \\ \underline{16.0} \\ 4 \end{array}$$

- "After we have divided 16 by 4, what is left to be divided?" (0.4.)
- Show this division in the algorithm:

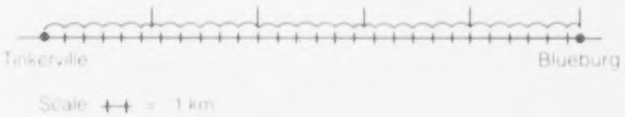
$$\begin{array}{r} 4.1 \\ 4\overline{)16.4} \\ \underline{16.0} \\ 4 \\ \underline{4} \\ 0 \end{array}$$

- Fold the calculator tape into 4 congruent parts. Measure 1 part to show that it is 4.1 meters long.
- Repeat with other partitive situations.

Children should also have experiences with measurement situations:

ACTIVITY 12-10 Measurement Division

- "A relay team is to run a 27.5-kilometer cross-country race. Each member of the team will run 5.5 kilometers. How many team members are there?"
- A diagram is useful to illustrate this situation. Begin at Tinkerville and show the 5 legs of the race: The first leg is 5.5 units (kilometers) long; the second is 5.5 units; and so on until Blueburg is reached.



- The answer is represented by the 5 legs.

- Point out that repeated subtraction can be used to determine the answer:

$$\begin{array}{r} 27.5 \\ -\ 5.5 \\ \hline 22.0 \end{array} \qquad \begin{array}{r} 22.0 \\ -\ 5.5 \\ \hline 16.5 \end{array} \qquad \begin{array}{r} 16.5 \\ -\ 5.5 \\ \hline 11.0 \end{array} \qquad \begin{array}{r} 11.0 \\ -\ 5.5 \\ \hline 5.5 \end{array} \qquad \begin{array}{r} 5.5 \\ -5.5 \\ \hline 0 \end{array}$$

- Complete the algorithm:

$$\begin{array}{r} 5 \\ 5.5\overline{)27.5} \\ \underline{27.5} \\ 0 \end{array}$$

- Repeat with other measurement situations.

Placement of Decimal Point in Quotient

Understanding the following ideas helps children learn how to place the decimal point in quotients.

Estimation is a way to determine the number of decimal places in a quotient. To divide a pair of numbers, as in the sentence $2.36 \div 0.9 = \square$, children should think "2.36 is a little larger than 2, and 0.9 is a little smaller than 1. Since $2 \div 1 = 2$, 2.36 divided by 0.9 will be about 2." When the quotient is determined, children should place the decimal point so that a single digit is in the ones place. An exercise such as the following provides practice in estimating. Each sentence becomes a true statement only when the decimal point is correctly placed.

$$\begin{aligned} 69.3 \div 3 &= 231 \\ 811.8 \div 22 &= 369 \\ 20.74 \div 3.4 &= 61 \end{aligned}$$

$$\begin{array}{r} 4.98 \\ 6.3_{\wedge}\overline{)31.3_{\wedge}74} \\ \underline{252} \\ 617 \\ \underline{567} \\ 504 \\ \underline{504} \end{array}$$

Children who know that common fractions can be used to express division can rewrite a decimal fraction, such as $0.3\overline{)6.3}$, as a common fraction: $^{6.3}/_{0.3}$. When both numerator and denominator are multiplied by 10, it becomes $^{63}/_{3}$. This can then be rewritten in algorithm form $3\overline{)63}$ so the division can be completed. This is the mathematical justification for the caret method of placing the decimal point in a quotient. If both divisor and dividend are multiplied by 10 or a power of 10 so that the divisor is changed to a whole number, the decimal point in the quotient can be placed immediately above the decimal point in the changed dividend. This is illustrated in the margin; a caret (∧) indicates the new position of the decimal point in the divisor and dividend. Children should use this procedure after its rationale has been made clear to them.

Eventually children should be guided to make this generalization: The number of decimal places to the right of the ones place in a quotient is the difference between the number of places to the right of the ones place in the dividend and the number there are in the divisor. Once they have stated this generalization, have children compare it with the generalization about how to determine the number of decimal places in a product. They should observe that the operations indicated by the two generalizations are the inverse of each other.

DECIMAL FRACTIONS: PRACTICE AND USE

Computer programs provide both practice and application formats. *Fractions, Decimals, and Percent*[1] is a six-disk program that includes computation with both common and decimal fractions, work with percent, and word-problem exercises involving the 3 forms of representing fractional numbers. *Get to the Point*[2] encourages children to think about the values of decimal fractions and to use estimation while determining values. Three games provide variety while children practice addition, subtraction, multiplication, and division with decimal fractions. In *Decimals*,[3] children explore both common and decimal fractions as they throw darts at a balloon on a number line.

Children who have become accustomed to computing with a calculator will have little difficulty learning to compute with decimal fractions. The calculator activities that follow provide practice and application opportunities.

ACTIVITY 12-11

Reading Words for Decimal Fractions

Write each of these decimal number words and the decimal numerals on the chalkboard or duplicate them to distribute to your children:

---

[1] Society for Visual Education (SVE), 1345 Diversey Parkway, Chicago, IL 60614.

[2] Sunburst Communication Inc., 39 Washington Avenue, Pleasantville, NY 10570.

[3] Control Data Publishing Company, 8100 34th Avenue South, Minneapolis, MN 55440.

two and five tenths
four and three tenths
six and nine tenths
five and seven tenths
19.4

sixty-one and forty hundredths
seventy-five and sixty-seven hundredths
four and twenty-one hundredths
ninety-six and nine hundredths
237.37

Instruct the children to enter the decimal numbers for the words in the first list into calculators, pushing the plus key after each entry. The numbers will have been read and interpreted correctly if the sum equals the decimal number at the bottom. Have the children do the same with the numbers named by the words in the second column.

ACTIVITY 12-12

Placing Decimal Points

Write each of these multiplication and division examples on the chalkboard or duplicate them to distribute to your children:

$$\begin{array}{r} 21.3 \\ \times 4.8 \\ \hline 10224 \end{array} \qquad \begin{array}{r} 0.78 \\ \times 4.3 \\ \hline 3354 \end{array} \qquad \begin{array}{r} 0.789 \\ \times 26.3 \\ \hline 207507 \end{array} \qquad \begin{array}{r} 3.621 \\ \times 0.4 \\ \hline 14484 \end{array}$$

$$\begin{array}{r} 263 \\ 3\overline{)7.89} \end{array} \qquad \begin{array}{r} 654 \\ 3.6\overline{)235.44} \end{array} \qquad \begin{array}{r} 0051 \\ 9\overline{)0.459} \end{array} \qquad \begin{array}{r} 632 \\ 2.7\overline{)170.64} \end{array}$$

Instruct the children to use a process of rounding off and estimation to determine where to place the decimal point in each example. Have them check their estimates by computing the actual answers with calculators.

ACTIVITY 12-13

Checking Products

Duplicate these multiplication combinations and distribute them to your children:

0.4 × 9
0.12 × 30
1.0 × 36
9 × 0.04
120 × 0.03
18 × 0.02
3 × 1.2
2 × 1.8
4 × 0.9
0.6 × 0.6
72 × 0.5

3.6 × 1
1.8 × 2
0.04 × 0.9
0.3 × 12
6 × 0.6
40 × 0.09
60 × 0.6
0.05 × 720
18 × 0.2
36 × 0.1
360 × 0.01

Instruct the children to put check marks next to the combinations that they believe equal 3.6. Have them use calculators to check their work.

ACTIVITY 12-14

Batting Averages

Put this table of baseball statistics on the chalkboard or duplicate it to distribute to each child:

Player	Hits	At Bat	Average
Canseco	199	594	
Madlock	180	565	
Clark	178	578	
Davis, C.	177	591	
Carter	161	538	
Maldonado	163	545	

Have one of the children tell the class how a player's batting average is determined, or you tell them that it is found by dividing the number of hits by the official number of times at bat.

Have the children use calculators to determine the batting averages of the six players listed in the table.

Extend this activity into a project by organizing the class into teams of 2 children each. Allow each team to select a major league baseball team, or you assign a team to each pair of children. Have the children determine the current batting average for each of their team's players. Have them maintain a chart on which they update each player's average once a week. Have each pair of children prepare a graph on which they show the weekly changes in their players' averages.

Textbooks for intermediate grades provide exercises for paper-and-pencil practice. These pages can be used if such practice is deemed important for a particular group of children. The "Rainbow Spectra" preprogrammed device[4] has cards dealing with introductory and practice exercises with decimal fractions.

You will also want to include many practical applications of decimal fractions outside of mathematics lessons. As children learn about the metric system, they will have opportunities to deal with decimal fractions. (Activities for learning about measurement are presented in Chapter 13. Science lessons frequently provide opportunities for dealing with decimal fractions. Many science situations involve measurement, so it is natural to integrate them with activities designed to help children learn about the metric system and its measuring devices. Children should be encouraged to find newspaper and magazine articles that show how decimal fractions are used in everyday situations. Bulletin board displays or notebooks can be used to organize their findings.

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[4] Educational Insights, 19560 S. Rancho Way, Dominguez Hills, CA 90220

PERCENT

In a sense, activities with percent in the sixth grade are a capstone for the work with comparisons that began in the first year of school. A part of children's first activities in mathematics is concerned with comparing sets to determine which one is larger or smaller or whether they are equivalent, and with comparing the length of 2 objects to find out which is longer or shorter or whether they are the same length. Later, children count discrete objects, measure continuous objects, and use subtraction to compare the sizes of sets or measures of objects. As their knowledge of mathematics increases, they use common and decimal fractions for making comparisons. Work with percent gives children yet another way of dealing with comparisons.

Meaning of Percent

When percent is used, the comparison is expressed as a ratio between some number and 100. For example, when 15 is used as a percent, it is an expression of the ratio between the numbers 15 and 100; it is symbolized as 15%. The symbol % expresses a denominator of 100; the word percent names the symbol and means *per hundred*, or *out of 100*.

In practice, percent is used in various ways, and its applications in business, government, science, industry, and other aspects of our lives are numerous. For example, a store reports that 16 percent of its sales are made in its garden department. This indicates that for every 100 dollars of sales, 16 are in the garden department. The complete report will contain information that tells the portion of sales made in the store's other departments, with the total amount of sales equaling 100 percent.

Some states use a sales tax as a means of raising revenue for part of their annual budget. A tax of 5 percent means that for every 100 cents (one dollar) of the purchase price of an article, a tax of five cents (0.05 of a dollar) must be collected. This situation provides the setting for a discussion of terms used in connection with percent and consideration of a point of confusion that often arises during the study of percent.

For a number of years the state of California used a 5-percent sales tax as a means of collecting a part of the money needed for its budget. When a $10.00 sale was made, 50 cents was collected for the sales tax. Three words are used to describe the different numerals in this situation. The 5 percent is the rate, $10.00 is the base, and 50¢ is the percentage.

When children encounter a statement such as the following, which might appear in a social studies textbook, they need to know how to interpret it so it will have meaning: "The imports of grain are 36 percent of the country's imports. The total value of all imports for one year was $236,000,000." In this statement, 36 percent is the rate and $236,000,000 is the base. Children must analyze the statement by thinking somewhat as follows: "I know that 36 percent represents a rate of $36 out of every $100. This means that about ⅓ of all the money value of the imports is for grain. One-third of $236,000,000 is about $80,000,000. The country imports about $80,000,000 worth of grain." Thus $80,000,000 is the estimated percentage of grain among all imports.

Percent and *percentage* are often confused, and many children (and adults, as well) are uncertain about the distinction between them. Because confusion is likely, you must make it clear that percent indicates the *rate* (of sales in a store's department, or of sales tax), whereas percentage indicates the *quantity*, or *amount* (of sales or of tax). Thus a base and a percentage always represent numbers that refer to the same units, for example, numbers of dollars of sales made or the weights of different ingredients in a mixture; and percent is the rate by which the percentage compares with the base. The potential for confusion is great enough to justify delayed use of the word *percentage* during elementary school, leaving it for teachers in higher grades to introduce. Its omission will not interfere with children's understanding of the basic ideas involved.

The California state sales tax provides the basis for discussion of another potential point of confusion. In the early 1970s, the tax rate was 5 percent. On the first of July 1973, the sales tax rate was changed from 5 percent to 6 percent. This was a very unpopular change, because at the time the state's treasury had a budget surplus of nearly $600,000,000. Later the legislature passed and the governor signed into law a bill that rescinded the increase for 6 months beginning October 1, 1973. Was the *rate* (percent) of decrease the same as the rate of increase? In both cases the *amount* was the same. The increase was 1 cent for every dollar spent, and the decrease was also 1 cent for every dollar spent. However, the rate of decrease was not the same as the rate of increase. The number-line graphs in Figure 12-9 show why the rates in this increase-decrease situation are different, even though the amounts are the same. In (a) the original sales tax, which was 5 cents for every dollar spent (5 percent), is represented by the first line and is shown as 100 percent, because 5 cents was the whole amount of tax for each dollar before the increase. The second line in (a) shows the sales tax after it was

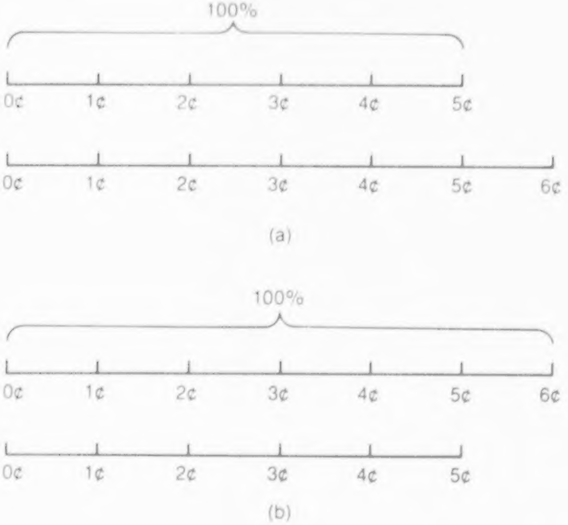


FIGURE 12-9

Number lines showing: (a) When a sales tax is increased from 5¢ to 6¢, the rate of increase is 20 percent. (b) When the tax is decreased from 6¢ to 5¢, the rate of decrease is 16⅔ percent.

changed to 6 percent. The amount of increase in the length of this line is the same as the length of 1 of the 5 parts of the original line. One of 5 is 20 percent of the whole; so the rate of increase was 20 percent.

In (b) the new sales tax is represented by the top line. It now represents 100 percent, because the rate was 6 cents for every dollar spent. The amount of decrease in the length of the second line is the same as the length of 1 of the 6 parts of the top line. One of six is 16⅔ percent of the whole; so the rate of decrease was 16⅔ percent.

Introducing Percent

Before children begin work with percent, they must have a clear understanding of fractional numbers and decimal-fraction representations of them because of the relationship between decimal hundredths and percent. There are many elementary-school children who fail to understand their work with percent, because they lack the necessary background. Do not start activities with youngsters whose backgrounds are inadequate, even if it means delaying their work until after their elementary-school years.

You can introduce percent to children who are ready through activities with learning aids used for other topics. The key point to make during early work is that *percent* means *per 100*, or *out of 100;* so it is necessary to use materials that show a unit, or whole, that can be readily subdivided into 100 parts or easily compared with that many parts. Especially helpful are magnetic materials, such as sets of poker chips with magnets attached, a hundreds board (such as the one in Activity 7-12), structured materials, and squared-paper decimal fraction kits.

In most cases you will introduce percent to the mathematically mature children who are ready for it and not to the entire class. Therefore problem cards (Activity 12-15) are a useful means of guiding these children's work.

ACTIVITY 12-15 Introducing Percent

Give each individual a flat, 5 orange rods, and 100 white cubes from a Cuisenaire set. Children work alone or in small groups to complete the work on a set of problem cards (pp. 467–469); answers are recorded on a response sheet (p. 470).

PERCENT (1)

Use your flat, orange rods, and white cubes to do these activities:

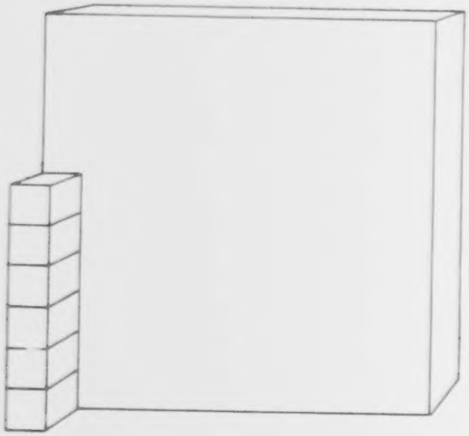
1. Lay your flat on your desk. Put 5 orange rods side by side on top of it. Write a statement on your response sheet using a common fraction to tell how much of the flat is covered by the 5 rods. Write a decimal fraction that tells how much of the flat is covered by the rods.

2. Cover the rest of the flat with white cubes. Count the white cubes, and then write a statement on your response sheet telling how many there are. Write another statement using a common fraction to tell what part of your flat is covered by the white cubes. Write a decimal fraction that tells how much of the flat is covered by white cubes.

3. Cover your flat completely with white cubes. Write a statement that tells how many white cubes cover one flat.

4. Some sets of white cubes are listed below. Use one set at a time, and put the cubes on your flat; then use both a common and a decimal fraction to tell the part of the flat that is covered by each set. The picture on the next card shows how to do the first one.
   a. 6 cubes
   b. 15 cubes
   c. 23 cubes
   d. 45 cubes
   e. 63 cubes
   f. 94 cubes
   g. 36 cubes
   h. 1 cube

(2)



Six cubes cover 6/100, or 0.06, of the flat.

5. So far you have used common and decimal fractions to tell what part of the flat is covered by rods and cubes. There is another way you can write your answers. You will learn about percent so you can use this new way.

In problem 1 you put 5 orange rods on the flat. Your statement with a common fraction could have used either 1/2 or 5/10 to tell how much of the flat was covered by orange rods. Your decimal answer could have been either 0.5 or 0.50.

(3)

In problem 2 you put 50 white cubes on the rest of the flat. Your statement with common fractions is correct if you used 1/2, 5/10, or 50/100 in it. Your decimal fraction can be either 0.5 or 0.50.

Another way to tell how many orange rods or white cubes cover half of the flat is with percent. Write 50 percent on your response sheet in the space you left blank earlier.

6. In problem 3 you used 100 white cubes to cover the flat. These cubes cover 100 percent of the flat. One hundred percent means that all of the flat was covered by white cubes. What do you think 50 percent means? Write your answer on your response sheet.

7. In problem 4 you used eight different sets of cubes and wrote common and decimal fractions to tell what part of the flat was covered by each set. The answer for the set of 6 cubes was written as 6/100 and 0.06. Another way to answer is to say that the cubes cover 6 percent of the flat.

   Use the rest of the spaces in problem 4 to tell what percent of the flat is covered by each of the sets of cubes.

When children have completed activities such as the ones on the problem cards, discuss the meaning of the word *percent* and the % symbol. Follow-up activities, such as naming the percent of the blue, red, and white chips in sets of 100 poker chips, are useful. (After children have used poker chips for 3 or 4 examples, you can give them pictured sets for practice.) Other activities include naming the percents of numbers blocked off on a hundreds board, and naming the percents of shaded or colored squares in 10-by-10 arrays. A variation of these activities that children enjoy is one in

PERCENT RESPONSE SHEET

Name ____________________

1. ______________________________
____
______________ Decimal fraction ______

2. ______________________________
____
____
______________ Decimal fraction ______
______________ (Leave this line blank for now.)

3. ______________________________
____

4. a. ___ ___ e. ___ ___
b. ___ ___ f. ___ ___
c. ___ ___ g. ___ ___
d. ___ ___ h. ___ ___

(Leave these lines blank for now.)

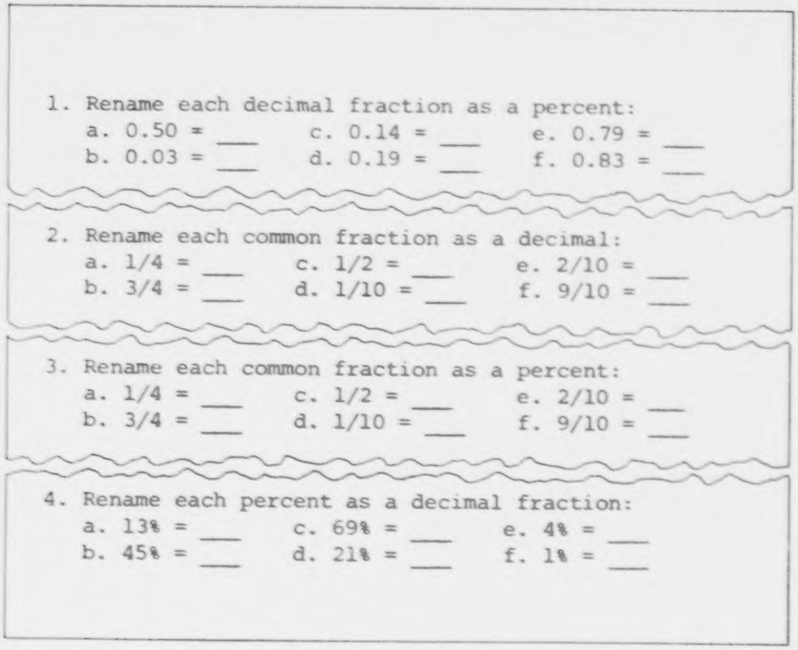
______ ______
______ ______
______ ______
______ ______

____
____
____

which they make colored designs in 100-square sections of squared paper. Children can mark off their own blocks of 100 squares. The blocks do not need to be 10-by-10 arrays but can be any arrangement that contains 100 squares. Have each child leave enough space between blocks to write the percents for the different colors used in each of the designs.

Extending Understanding of Percent

The problem cards in Activity 12-15 contain activities that acquaint children with the idea that percent is an extension of the common and decimal-fraction forms of representing comparisons. Subsequent work with poker



1. Rename each decimal fraction as a percent:
   a. 0.50 = ___   c. 0.14 = ___   e. 0.79 = ___
   b. 0.03 = ___   d. 0.19 = ___   f. 0.83 = ___

2. Rename each common fraction as a decimal:
   a. 1/4 = ___   c. 1/2 = ___   e. 2/10 = ___
   b. 3/4 = ___   d. 1/10 = ___   f. 9/10 = ___

3. Rename each common fraction as a percent:
   a. 1/4 = ___   c. 1/2 = ___   e. 2/10 = ___
   b. 3/4 = ___   d. 1/10 = ___   f. 9/10 = ___

4. Rename each percent as a decimal fraction:
   a. 13% = ___   c. 69% = ___   e. 4% = ___
   b. 45% = ___   d. 21% = ___   f. 1% = ___

FIGURE 12-10

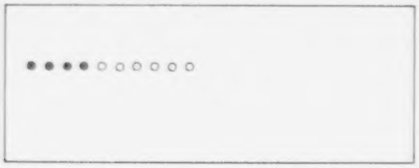
Sample activity sheet for renaming common and decimal fractions and percents

chips, hundreds boards, and other manipulative and pictorial materials should continue to emphasize the relationships among common and decimal fractions and percents. Once the relationships are clear, children are ready to move to more abstract levels of work. They can begin to name the equivalents for different fractions that are given in one or another of the 3 numerals. An activity sheet that contains several sections can be prepared. Each section has expressions of 1 type, which are to be changed to another form (Figure 12-10).

Not all percents arise from situations in which 100 is conveniently present as the quantity against which another number is compared, as in the introductory situations given above. Therefore, once the basic concept is well established, children's work should focus on activities that deal with groups other than 100. For example, if you begin with a set of 10 chips, 4 of which are blue and 6 white, what percent of the entire set is the group of blue chips?

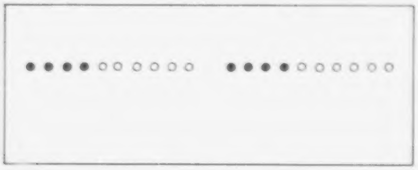
ACTIVITY 12-16 Percent

- Begin by showing 10 magnetic chips, 4 blue and 6 white, on a magnetic board.



Note that there are a total of 10 chips. "What part of the set is blue?"($\frac{4}{10}$ or 0.4.) "Can anyone tell what percent of the chips is blue?" (40%). If a child gives the answer, ask for an explanation of how it was determined. If no answer is given, ask, "What must we do to change $\frac{4}{10}$ to a fraction with a denominator of 100?" (Multiply both numerator and denominator by 10.) "When we do this, what fraction is equivalent to $\frac{4}{10}$?" ($\frac{40}{100}$.) "$\frac{40}{100}$ is what percent?" (40.)

Show a set that has 8 blue chips out of 20. "Are the blue chips in this set 40% of the set?" (Yes.) Discuss what is done to confirm the answer. (Multiply both numerator and denominator by 5.)



Repeat with other sets that have the same ratio of blue chips to total chips.

Complete a table with other fractions that are equivalent to $\frac{4}{10}$ to show that in each case the blue chips are 40% of all of the chips.

4	8	12	16	20	24	28	32	36	40
10	20	30	40	50	60	70	80	90	100

Repeat with other sets wherein the total number of chips is a divisor of 100 to make the renaming of fractions with denominators of 100 easy.

Use whatever additional activities are necessary to make certain that children can find what percent 1 number is of another in situations in which the comparison is made with both 100 and numbers other than 100. When numbers other than 100 are used, they should be those for which the equivalent ratio containing 100 can be easily determined. A ratio of 4 out of 10 is suitable, but a ratio of 7 out of 13 is not. In the elementary school, percents less than 1 and greater than 100 are seldom used.

Working Percent Problems

Children who successfully complete the activities already discussed are ready for percent problems. There are 3 methods of solving percent problems: the proportion method, the case method, and the unitary analysis method. The proportion method is used most widely in elementary-school programs and is the only one included in this book.

The proportion method has gained favor in recent years because it is easier for most children to learn and use. It is based on the idea that a single expression can be used to show each of the 3 types of percent problems. Children must understand the meanings of percent (rate), percentage, and base and how to express these as a proportion. The proportion expression is

$$\frac{\text{rate}}{100} = \frac{\text{percentage}}{\text{base}}$$

Children will have encountered other proportion expressions if they have completed activities such as those described in Chapter 11, where tables are developed and markers used to illustrate proportion situations. A parking lot that contains 50 cars, with 18 of the cars foreign-made, will serve as a setting for considering the proportion method of solving percent problems. The 18 foreign cars are 36 percent of the total number of cars.

When the percent and the total number of cars are known, we can determine the percentage by putting the 2 known terms in the proportion expression and solving for the percentage.

$$\frac{36}{100} = \frac{\text{percentage}}{50}$$

When the number of foreign-made cars and the total number of cars are known, fill in the expression with the 2 known terms and solve for the unknown term:

$$\frac{\text{percent}}{100} = \frac{18}{50}$$

In the third situation, the percent and percentage are known, and the base is to be determined; so the expression becomes

$$\frac{36}{100} = \frac{18}{\text{base}}$$

In the examples used here, the solution is not difficult. For $^{36}/_{100} = n/50$, children will know that 100 is divided by 2 to give 50; so they know that 36 divided by 2 will give the answer, 18. For $n/100 = {}^{18}/_{50}$, they will know that

2 times 50 is 100; so the answer can be found by multiplying 2 times 18. And for $36/100 = 18/n$, 36 is divided by 2 to give 18; so 100 is divided by 2 to give 50.

Not all proportion problems can be solved as easily as these. Therefore children should become familiar with the procedures explained in Chapter 11, so they can use the more mature methods to solve all types of percent problems, as well as the simple ones used here.

Summary

The fractional number system includes all of the positive and negative integers and numbers between integers that can be expressed in the form a/b, where a is any integer and b is any integer other than 0. Fractional numbers can be expressed with common fractions, decimal fractions, and percent. Fractional numbers greater than 0 are included in the elementary curriculum. Ways to introduce decimal and percent representations are presented in this chapter, along with ways of teaching operations with decimal fractions.

Work with decimal fractions and percent extends the understanding of fractional numbers and ways to express them. Decimal fractions express numbers smaller than 1 that have a denominator of 10 or a power of 10. The denominator is not written but is indicated by how many numerals there are to the right of the ones place in a decimal numeral. Structured materials, unit-region kits, and number lines are aids children use to learn the meaning of decimal fractions. Once the meaning of decimal fractions is understood and children can write decimal numerals accurately, they are ready to learn to add, subtract, multiply, and divide fractional numbers expressed as decimals. The algorithms for these operations are like those for whole numbers. Real-life situations and work with learning aids will help children extend their use of the algorithms to include decimals. Carefully developed procedures for determining where to put the decimal points in answers should be used rather than teacher-dictated rules. Computer programs provide both developmental and application activities. Calculators should be used for routine computation and practice. Textbooks and preprogrammed machines can be used for practice exercises. Applications of decimal fractions in science and measurement situations should be integrated to help children learn how they are used for practical reasons.

Percent is used to express a ratio between some number and 100. Carefully sequenced activities will help mature elementary-school children understand its meaning. Special attention needs to be given to the meanings of percent (rate), base, and percentage. The proportion method of solving percent problems is described in this book. Instruction about it can be related to work with rate pairs and proportion completed during earlier problem-solving activities.

Study Questions and Activities

1. Estimation is recommended as a way to help children learn how to place a decimal point in the product or quotient for computation with decimal-fraction notation. Explain the thought process a child might use to estimate a product or quotient for each of these examples:

$$\text{(a)}\ \begin{array}{r} 4.8 \\ \underline{\times 2.3} \\ 1104 \end{array} \qquad \text{(b)}\ \begin{array}{r} 36.2 \\ \underline{\times 1.08} \\ 39096 \end{array} \qquad \text{(c)}\ \begin{array}{r} 48.36 \\ \underline{\times 51.28} \\ 24799008 \end{array} \qquad \text{(d)}\ \begin{array}{r} 682 \\ 3.2\overline{)21.824} \end{array} \qquad \text{(e)}\ \begin{array}{r} 302 \\ 0.33\overline{)9.966} \end{array}$$

Explain why estimation is not useful in examples like these:

$$\text{(a)}\ \begin{array}{r} 0.341 \\ \underline{\times 0.682} \\ 232562 \end{array} \qquad \text{(b)}\ \begin{array}{r} 2 \\ 0.036\overline{)0.0072} \end{array}$$

Describe a process for placing the decimal point in each of these 2 examples.

2. Dewar explains a model for solving problems involving percent (see For Further Reading section). Demonstrate through drawings and written explanation how that model could be used to solve these problems:
   (a) A mixture of coffee beans weighs 50 kilograms. If 48 percent of the beans are grown in Hawaii, how many kilograms of the coffee are from Hawaii?
   (b) A mixture of coffee beans contains 24 kilograms of coffee grown in Hawaii. If this is 48 percent of the total mixture, what is the total weight of the mixture?
   (c) A mixture of coffee that weighs 50 kilograms has 24 kilograms of coffee grown in Hawaii. What percent of the mixture is coffee from Hawaii?

3. Carpenter and his associates (see For Further Reading section) evaluated children's work on the decimal-fraction portion of the *National Assessment of Educational Progress* test and identified weaknesses. They indicated the implications of the test results for teachers. What do they say is needed to improve instruction about decimal fractions? Describe materials you believe should be available so children can overcome the weaknesses identified by these authors.

For Further Reading

Carpenter, Thomas P., et al. "Decimals: Results and Implications from National Assessment," *Arithmetic Teacher*, XXVIII, No. 8 (April 1981), pp. 34–37. The latest NAEP test indicates that children lack a clear understanding of decimal fractions. This and other weaknesses are reported, along with implications of the test's results for classroom teachers.

Cole, Blaine L., and Henry S. Weissenfluh. "An Analysis of Teaching Percentage," *Arithmetic Teacher*, XXI, No. 3 (March 1974), pp. 226–228. Describes the relationship between the ratio method and the case method of solving percent problems. Explains the ratio method and gives a brief outline of the background for understanding it.

Dewar, Jacqueline M. "Another Look at the Teaching of Percent," *Arithmetic Teacher*, XXXI, No. 7 (March 1984), pp. 48–49. The author explains how a simple drawing can be used to illustrate the meaning of and operations dealing with percent. Examples of different types of problems are used to demonstrate how the drawing works.

Firl, Donald H. "Fractions, Decimals, and Their Futures," *Arithmetic Teacher*, XXIV, No. 3 (March 1977), pp. 238–240. The author predicts an increasing emphasis on decimal fractions, because of the advent of the calculator and greater use of the metric system. The role of common fractions will diminish, but they cannot be eliminated entirely. Even after total metrication, there will be a continuing need for certain common fractions.

Fisher, John W. "Deci-Deck," *Arithmetic Teacher*, XXII, No. 2 (February 1975), p. 149. The deci-deck card game provides practice in naming equivalent common fractions, decimal fractions, and percent. Several card games are possible with the deck.

Lightenberg, Betty K., and Donovan R. Lichtenberg. "Decimals Deserve Distinction," in *Mathematics for the Middle Grades (5–9)*, 1982 Yearbook, National Council of Teachers of Mathematics (Reston, Va.: The Council), pp. 142–152. Materials and procedures for teaching decimal fractions to fifth-grade and older children are illustrated and described. Processes for comparing and rounding decimal fractions are included.

Priester, Sharon. "SUM 9.9: A Game for Decimals," *Arithmetic Teacher*, XXXI, No. 7 (March 1984), pp. 46–47. A card game—ONO 99—served as the inspiration for SUM 9.9, which is a game for practicing addition and subtraction with decimal fractions. The card deck and rules are explained.

Vance, James H. "An Opinion Poll: A Percent Activity for All Students," in *Mathematics for the Middle Grades (5–9)*, 1982 Yearbook, National Council of Teachers of Mathematics (Reston, Va.: The Council), pp. 166–171. Children work in pairs to poll individuals on their preference for foods, cars, sports, and so forth. They compile and organize their data, using charts and graphs. Percents are used to compare preferences in each chart.

Chapter

13

Measures and the Processes of Measuring

In Chapter 13, you will read about

1 the meaning of measurement and two measuring processes

2 the history of the metric system and its basic characteristics and advantages over the English (customary) system

3 the role of children's readiness in planning instruction for measurement concepts

4 activities and procedures for building a foundation for measurement

5 a sequence of linear measurement activities with nonstandard and standard units

6 commonly used units of weight (mass) measures and two types of scales for children

7 area measure and activities for learning to measure the area of plane figures

8 volume measure and procedures to use in determining volume of space figures such as cubes and rectangular prisms

9 the meaning of angular measure and a process for helping children learn to measure angles

10 setting up a metric mini-Olympics

11 developmental activities for time and its measures

12 activities for understanding and using money for exchange

On December 23, 1975, President Gerald Ford signed the Metric Conversion Act of 1975. This act (Public Law 94-198) was designed to change the official system of measure in the United States from the English (customary) to the metric system. Changeover was not mandated by this act; it was to be voluntary. Even so, it was anticipated that business and industry, governmental agencies, and the population in general would accept the plan so that conversion would be complete within a few years. The Federal Highway Administration planned to change distance markers to kilometers and speed limits to kilometers per hour by July 1, 1978. The National Weather Service planned to report temperatures, wind speed, precipitation, and air pressure in metric units by January 1, 1979. Companies in the private sector, such as Black and Decker, IBM, Coca-Cola and Seven-Up, Butterick and Simplicity, and Chrysler, Ford, and General Motors, changed entirely or partially to the metric system. It seemed that the United States, the last major nation not yet metricated, would join other countries of the world as a metric nation.

The hopes and expectations of those who supported the move to metrication have not materialized. Public resistance was great enough to thwart plans to indicate distances, speed limits, and weather information in metric units. The U.S. General Accounting Office reported that the benefits may not be as great as were anticipated and recommended that immediate conversion not be undertaken. On September 30, 1982, the United States Metric Board, established in 1975 to oversee the conversion process and provide guidance to governmental, business, industrial, and educational institutions, was disbanded. Metrication in the private sector has not continued at the pace seen in the years immediately following passage of the conversion act.

The slowdown of conversion plans means that the customary system of measures will continue as the predominant system in the United States for the foreseeable future. Changes already in place will probably remain, but further large-scale changes are unlikely for the present time. Therefore children need to develop an understanding of both the metric and the customary systems and to become skillful users of measuring instruments for both systems.

This chapter includes a discussion of the meaning of measurement, a brief history of the metric system, the metric system's characteristics, and examples of activities for teaching about measures and measuring processes.

The metric system serves as the example for each activity. However, the customary system is taught in identical ways, so its units could be substituted wherever metric units are used.

WHAT IS MEASUREMENT?

The distinction between discrete and continuous objects is made in Chapter 6, in which children's early mathematical experiences are discussed. Discrete objects are those that can be counted, whereas continuous objects are measured. Measurement is the process of attaching numbers to the physical qualities of length, area, volume, weight (mass), or temperature. (Time is also measured, but it lacks a physical quality.) Each physical quality can be measured by means of 1 or more units. For example, the meter[1] is the basic unit for referring to an object's length, but units larger and smaller than the meter are used for finding the length of objects for which the meter is inappropriate.

Two Types of Measuring Processes

Two types of measuring processes exist: direct measure and indirect measure. The processes for determining most measures of length and capacity are direct, made by applying the appropriate unit directly to the object being measured. This direct process is referred to as iteration. Iteration is illustrated by the use of a meter stick to measure a room's length. The length is measured by placing the zero end of the stick against 1 wall, marking a spot on the floor at the other end, moving the stick to put the zero end at the mark, and repeating the process until the opposite wall is reached. All the while, the number of times the stick is moved is counted. This number is the room's length expressed in meters.

Weight (mass), temperature, and time cannot be measured in the direct fashion just described. The characteristics, or properties, of weight, temperature, and time require an instrument that measures each one by translating the measurable property into numbers indirectly. Thus 1 type of weather thermometer has a number scale aligned with a tube containing a liquid. The liquid expands (and rises) when it is warmed as the surrounding air becomes hotter; it contracts (and goes down) when it is cooled as the air becomes colder. The temperature at a given time is determined indirectly by reading the numeral on the scale that indicates the height of the column of liquid at that time.

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[1] *Meter* is the accepted spelling in the United States for the basic unit of length, and *liter* is the spelling for the basic unit of capacity (volume). Children should also know the alternate spellings for these words—*metre* and *litre*.

Measurement Is Approximate

Although the counting of discrete objects is exact, all measurement is approximate. The approximate nature of measurement stems from the fact that for any particular unit of measure there is, theoretically at least, one that is more precise. If a city block is measured with a meter stick, it might be found to be 100 meters long. However, if a unit 1/10 of a meter—a decimeter—is used to measure the same block, it might be found to be 998 decimeters long, or slightly less than 100 meters. Even more precise measurements could be made using centimeters or millimeters, although in most instances this would be impractical and unnecessary.

THE METRIC SYSTEM

The metric system has assumed a more prominent role in the United States, in spite of setbacks in the conversion plan, and children must learn to understand and use it. This section gives a brief overview of the history of this system, discusses its key characteristics, and presents its advantages over the English (customary) system.

Brief History of the Metric System

The earliest proposal for a worldwide decimalized system of measure was made in France in the late 1600s, but a century passed before serious attention was given to such a system. Then in 1790, the National Assembly of France asked the French Academy of Sciences to devise a system having an invariable standard for all measures. The Academy derived its unit of length from a portion of the earth's circumference. This standard, which is called a meter, was one ten-millionth of the distance from the equator to the North Pole along the meridian running near Dunkirk in France and Barcelona in Spain and was represented by a prototype measure made of platinum.

Other units were derived from the meter (m). The units of linear measure are decimal parts or multiples of the meter based on powers of 10. A decimeter (dm) is 1/10 of a meter, a centimeter (cm) is 1/100 of a meter, and a

TABLE 13-1

Base units in the international system of units (SI)

Quality to be Measured	Unit	Symbol
Length	meter	m
Weight (mass)	kilogram	kg
Time	second	s
Electric current	ampere	A
Thermodynamic temperature	kelvin	K*
Amount of substance	mole	mol**
Luminous intensity	candela	cd

* Degrees Celsius is an acceptable alternate unit for measuring temperature.

** Some listings contain only 6 units, omitting the mole. See M. J. B. Jones, *A Guide to Metrication* (Oxford, England: Pergamon Press Ltd., 1969), p. 15.

millimeter (mm) is $\frac{1}{1000}$ of a meter. A dekameter (dam) is 10 meters, a hectometer (hm) is 100 meters, and a kilometer (km) is 1000 meters.[2]

Basic units for capacity (volume), weight (mass), and area were established at the time the base unit for linear measure was established. A liter (L), the basic unit for capacity, is the volume of 1 cubic decimeter (dm^3). A kilogram (kg) is the basic unit of weight (mass) and was originally determined to be 1 cubic decimeter of water at 4°C, the temperature at which water is at its greatest density. Today's international prototype of the kilogram is a platinum–iridium solid maintained at Sèvres, France. For measuring land, the basic unit is the are (a), which is 100 square meters (m^2). A hectare (ha) is 100 ares. Volume is also measured in cubic meters (m^3) and cubic centimeters (cm^3).

In 1960 the Eleventh General Conference of Weights and Measures adopted the International System of Units (SI). There are 7 basic units and 2 supplementary units, as shown in Tables 13-1 and 13-2. The SI system is accepted worldwide with nearly full agreement among nations on the units, their values, names, and symbols. An exception to full agreement is the use of L as the symbol for liter in the United States, whereas l is the SI symbol.[3]

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[2] There are smaller subdivisions and larger multiples of the metric system's basic units, but they are seldom used outside the field of science and are not included in the elementary-school curriculum.

[3] Use of the capital L in the United States as the symbol for liter avoids the possibility of confusing the symbol with the numeral 1. The use of the script *l* was discarded, because of its unavailability on typewriters and other printing devices at the time the decision was made.

TABLE 13-2
Supplementary SI units

Quality to be Measured	Unit	Symbol
Plane angle	radian	rad
Solid angle	steradian	sr

One detail of interest coming from the 1960 conference was a redefinition of the meter. It is approximately the same length as the original meter but is now defined as 1 650 763.73 wavelengths[4] in vacuum of the orange-red line of the spectrum of the krypton-86 atom. This standard was adopted because it is unvarying and indestructible and can be reproduced in any laboratory possessing the proper equipment.

The SI system is highly technical, and only the parts dealing with length, mass, and temperature and certain non-SI metric units dealing with time, area, and volume are studied by elementary-school children.

Characteristics of the Metric System

One significant characteristic of the metric system has already been mentioned—the fact that measures of capacity, weight, and area are all based on the meter.

Another important feature is that the metric system is a decimal system. This means that for each type of measure there is an interrelationship of units based on multiplication or division by 10. This scheme is essentially the same as the Hindu-Arabic numeration system, a fact that should be capitalized on to help children develop their understanding of the metric system. Table 13-3 shows the relationship between the metric system and the Hindu–Arabic numeration system and certain portions of our monetary system. Only certain of the measures shown in Table 13-3 are commonly used: the meter, kilometer, centimeter, and millimeter; gram, kilogram, and milligram; and liter and milliliter. These, along with certain units for area (square meter and square centimeter), volume (cubic meter and cubic centimeter), time (year, day, hour, minute, and second), and temperature (degrees Celsius), are the units most elementary-school children will use during their study of the metric system and are the units considered in this book.

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[4] The comma is omitted from numerals for large and small numbers in SI. Large numbers appear as indicated here, and small ones appear as 0.362 45. This practice is followed because the comma serves as a decimal point in some countries.

TABLE 13-3

Relationship between parts of the metric system and the Hindu–Arabic numeration system and pertinent parts of our monetary system

	1000x **(10^3)**	**100x** **(10^2)**	**10x** **(10^1)**	**Base**	**$\frac{1}{10}x$** **(10^{-1})**	**$\frac{1}{100}x$** **(10^{-2})**	**$\frac{1}{1000}x$** **(10^{-3})**
Hindu–Arabic system	thousands	hundreds	tens	ones	tenths	hundredths	thousandths
Monetary system	$1000.00	$100.00	$10.00	$1.00	$0.10 (10¢)	$0.01 (1¢)	$0.001 (mill)
Metric prefix	kilo-	hecto-	deka-		deci-	centi-	milli-
Length	kilometer (km)	hectometer (hm)	dekameter (dam)	meter (m)	decimeter (dm)	centimeter (cm)	millimeter (mm)
Capacity	kiloliter (kL)	hectoliter (hL)	dekaliter (daL)	liter (L)	deciliter (dL)	centiliter (cL)	milliliter (mL)
Weight (mass)	kilogram* (kg)	hectogram (hg)	dekagram (dag)	gram (g)	decigram (dg)	centigram (cg)	milligram (mg)

* The kilogram is the basic unit for weight. Therefore it is a contradiction of the idea that the base units stand without a prefix.

Advantages of the Metric System

The following advantages of the metric system are reported in "1 . . . to get ready"[5]:

1. The metric system is simple and logically planned and is a decimal system like our numeration and monetary systems.
2. The basic unit, the meter, is always reproducible from natural phenomena.
3. A single set of prefixes is used to designate all subdivisions and multiples of units for all types of measures. This simplifies the process of converting from 1 unit to another.
4. Uses of common fractions will be reduced, which means that less time will be required for children to understand the common fractions they will need. The time gained can be devoted to other important topics.
5. Familiarity with the meter, gram, and liter and their subdivisions and multiples will be sufficient for most persons. Extension of the study of the International System of Units can be made by those persons who need a deeper understanding of the system.
6. The metric system is the universal measuring language.

CHILDREN'S READINESS FOR MEASUREMENT ACTIVITIES

Piaget's research and its general implications for teaching mathematics are discussed briefly in Chapter 1. The section dealing with this research should be reviewed now, because the contents of that discussion are important in the planning of measurement activities. That discussion is now extended to introduce additional information dealing with children and conservation.

To study children's ability to conserve length, a pair of rods or sticks of equal length can be used. Initially, place the rods side by side on a table with their ends an equal distance from 1 edge, as in Figure 13-1(a). Ask the child if the rods are of equal length. A child who understands the question will usually agree that they are. Next, move 1 stick so it is farther from the table's edge (b), and repeat the question. A child who is not a conserver of length will say that the sticks no longer have equal lengths. A conserver

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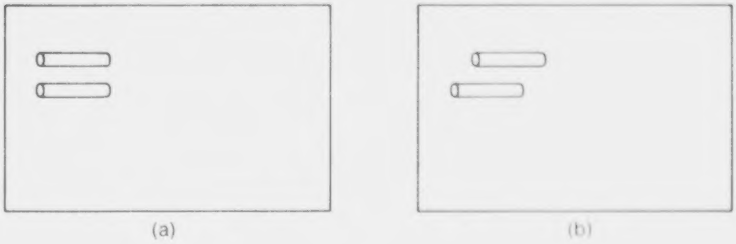
[5] "1 . . . to get ready" is a selected bibliography on metrication issued in April 1973 by a Joint Committee of the American Association of School Librarians and the National Council of Teachers of Mathematics.

FIGURE 13-1

Test for determining a child's ability to conserve length



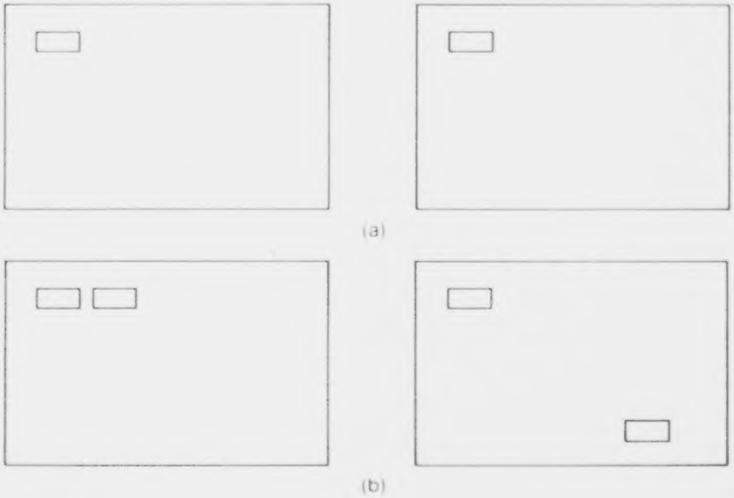
recognizes that the length of either rod is not altered by changing the position of 1 of them. Generally children are unable to conserve length until they are about eight years of age.

A similar test can be used to determine whether a child is a conserver of area. A pair of "fields," represented by equal-sized pieces of green paper, and "buildings," represented by smaller equal-sized pieces of brown paper, are used. Place the fields side by side with a building in place on each, as in Figure 13-2(a). Ask the child if 1 building covers the same amount of land as the other. Next, place an additional building on each field, as shown in (b), and rephrase the question. A nonconserver will say that the buildings on 1 field cover more land than those on the other. The conserver recognizes that an equal number of equal-sized buildings cover the same amount of land (area) regardless of their positions. Children generally do not conserve area until around eight years of age.

Neither of these tests alone is sufficient to determine whether a child is a conserver of area or length. Several tests using similar materials and tech-

FIGURE 13-2

Test for determining a child's ability to conserve area



niques should be used before a firm decision about a child's status can be reached.[6]

Children's reactions to these and similar tests give curriculum planners and teachers a basis for selecting areas of measurement and their sequence for children to learn about in the elementary school. It is generally recommended that children's early experiences be exploratory, with ample free-play activities dealing with pouring, matching, balancing, ordering, sorting, and comparing. In the primary grades, children typically deal with nonstandard units first, followed by activities with standard units. They learn to estimate and measure lengths, volumes, and weight. It is also in these grades that children learn about digital and traditional clocks to tell time, Celsius and Fahrenheit thermometers to measure temperature, and recognizing and counting money.

Practice with both metric and customary measuring instruments in the intermediate grades helps children become capable of choosing an appropriate unit of measure for length, volume, and weight. Intermediate-grade children learn to convert measures within each system from 1 unit to another. Processes and formulas for determining perimeter, area, and volume are also learned in the intermediate grades.

EXPLORATORY ACTIVITIES

Children who are still in the preoperational stage of development and who are not yet conservers should not be given activities dealing with standard units of measure. Rather they need a variety of activities that build a foundation for later work. In addition to those in the readiness section of Chapter 6, the following types of activities should be made available to young children:

ACTIVITY 13-1 Capacity

The purpose of this activity is to provide a background for understanding capacity.

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[6] For additional discussion of conservation of length and area, and examples of other tests, see Richard W. Copeland's *Math Activities for Children* (Columbus, Ohio: Charles E. Merrill Publishing Company, 1979), Ch. 4.

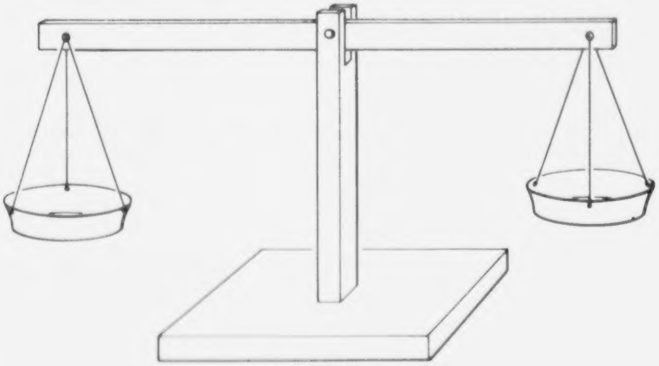
Prepare plastic or glass containers of various sizes by gluing rubber bands around them at different levels. Put 2 to 6 pounds of rice, a funnel, and a scoop in a plastic tub or large cardboard box.

Children use the scoop and funnel to fill each container to the rubber-band level.

This exploratory activity is not designed for children to count the number of scoops required to fill each container. However, some children may be able to do this, so provide squares of paper for them to record their counts. Encourage children to order the containers from least to most amount of rice.[7]

ACTIVITY 13-2 Heavier and Lighter

A balance beam, either commercial or teacher-made, gives children opportunities to compare pairs of objects to determine which is heavier or lighter.



Children's first work with a balance beam should be exploratory. They will learn to make it balance by putting an object in 1 pan and other objects in the second pan until the crossbar is level. They also see the crossbar go down at 1 end when the object in 1 pan is heavier than the object in the other pan.

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[7] Mary Baratta-Lorton, *Workjobs* (Menlo Park, Calif.: Addison–Wesley Publishing Company, 1972), pp. 22–23.

Set up pairs of objects—toy car and wooden block, chalkboard eraser and small bag of rice, small rubber ball and box of pencils—so children can determine which object is heavier and which is lighter. They can place the objects on a mat having the headings *lighter* and *heavier* separated by a line down the middle.

Lighter	Heavier

ACTIVITY 13-3

Passage of Time

This teacher-led activity gives children an experience in dealing with the passage of time.

Use 4 or 5 cans alike both in shape and size. Punch a hole in the bottom of each: a small hole in one, a large hole in another, and in-between sizes in the others. Paint each can a different color. Put water or clean, fine-grained sand in a plastic tub.

Have a child select a can, fill it, and hold it so the water or sand drains through the hole. Have children watch carefully as each can is filled and drained.

Ask questions about what they observe:

- "Does the red can empty quickly or slowly?"
- "Which can drains more slowly, the red one or the blue one?"
- "What is the color of the can that drains fastest?"
- "Can you put the cans in order from fastest to slowest?"
- "What can you tell me about the size of a hole and how fast a can drains?"

ACTIVITY 13-4

Temperature

Outdoors on a sunny day provides a setting for an investigation and discussion about temperature. Choose a location where there is a sidewalk, some grass, and a patch of bare ground, and select a time when 1 part of each is in sunlight and another part is in shade.

Have children remove their shoes and socks and walk on the sunny sidewalk. Talk about how it feels. Have them walk on the shady sidewalk and compare its feel with that of the sunny walk.

Have the children walk on the sunny and shady portions of the other surfaces. Discuss differences in temperature, if any, between each spot and the others. Can the children order the 6 locations from warmest to coolest?

NONSTANDARD UNITS

Children's understanding of measure begins as they engage in activities such as those already described. Continue with those activities until you see evidence that some children are able to conserve length, volume, and so on. After simple tests show which children are able to conserve length, for example, group them for their first work with nonstandard units.

Purposes of the Activities

Activities with nonstandard units bridge the gap between exploratory work and the introduction of standard units. Two goals of these activities are to help children recognize the need for a uniform set of measures and to lay the groundwork for their appreciation of the simplicity of the metric system.

Suggested Activities

The activities that follow are examples of those dealing with nonstandard units for the various types of measures.

For their first work with linear measure, children should use such things as new pencils for nonstandard units.

ACTIVITY 13-5 New-pencil Measurements

New pencils serve as units of measure in this introduction to linear measure. Select an object such as a desk top to measure. Demonstrate how to put pencils end to end to find the length and width of the desk in pencil units. Stress the importance of not leaving gaps or letting pencils overlap and of counting the pencils carefully.

- Use problem cards as a means of giving directions to individual and small group activities.

MEASURING WITH PENCILS

Use new pencils from the box to answer these questions.

1. How long is the reading table?
2. How wide is the reading table?
3. Can you figure out a way to find out how high the table is? If you can, tell how high it is.

Write each of your answers in a complete sentence.

When a single object, rather than many, is used to measure the length of an object, the iteration process is used. (Iteration is explained earlier in this chapter.) A child's cutout foot provides a useful nonstandard unit of measure for introducing the process of iteration.

ACTIVITY 13-6 Foot Units

- Have each child make an outline of his or her foot on a piece of paper and cut it out.
- Demonstrate how to use the cutout foot to measure the length and width of a desktop or tabletop.
- Give the children a list of objects to measure with their "foot" units, such as the length of the chalkboard chalk tray, the length and width of a file cabinet, and the width of the room. Add others of your own choosing.
- Children should record each measure in "foot" units.
- During a class discussion, talk about the children's measures for each object. When the number of units that are reported for a given object, such as a chalkboard tray, are not the same, stress that when nonstandard units are used, differences may occur because their "foot" units are not all the same length.
- Make a chart on which of the children's "foot" measures are ordered from shortest to longest.

An extension of exploratory activities with a balance beam provides experiences with nonstandard units for weighing objects.

ACTIVITY 13-7

Nonstandard Weights

Provide objects such as walnuts, wooden blocks, Unifix cubes, and small bags of rice as units of measure.

Use problem cards and answer sheets for children's responses. Questions like these can be used on problem cards:

- ☐ "How many Unifix cubes balance the red bag of sand?"
- ☐ "How many walnuts balance 1 block?"

Add other questions to fit your objects.

Rice and water- or sand-table activities can also be extended to include experiences with nonstandard units for measuring capacity.

ACTIVITY 13-8

Colored-can Units

Select cans so that one has 4 times the capacity of another, 1 has a capacity twice another, and so on. Paint all cans with the same capacity a given color.

Direct children's activities with problem-card questions:

- ☐ "How many blue cans does it take to fill 1 red can?"
- ☐ "How many green cans can you fill from 1 yellow can?"

Prepare other questions based on the cans at your table.

Glass sand-filled timers can be used as nonstandard time-measuring units. Get as many different timers as possible. Children can observe that the sand transfers more quickly from one than from another and that the sand in 2 differently sized timers can transfer in the same amount of time.

Squares cut from corrugated boxes can be used to introduce children to the concept of area. The idea to be developed during early activities is that area is the measure of a surface enclosed by a closed plane figure. Under your direction, children can cover desktops and tabletops, sections of carpet, and other surfaces with cardboard squares and count the squares covering each one. Answers should be labeled as "square units of cardboard."

STANDARD UNITS OF MEASURE

It has already been said that children need to learn both the metric and the customary systems of measure. The premise of this book is that children learn about measurement best when they have frequent opportunities to measure objects. Rather than reading about a measuring process, such as linear measure, children should use an appropriate measuring instrument to actually measure objects. As they work, they learn that there are 10 decimeters in a meter or 12 inches in a foot. Once children are familiar with units for 1 system, they can learn the comparable units for the other system.

The time spent on learning both systems need not be extensive. Children who understand the basic concepts of measure and who learn to use measuring instruments properly can learn units of measure in both systems simultaneously.

It is recommended that children not be asked to convert measurements from customary units to metric units and vice versa. Beyond learning that a meter is slightly longer than a yard, a liter is a little more than a quart, and a kilogram is a little more than 2.2 pounds, there is little to be gained from having children compare units in the 2 systems. Heavy doses of conversions weaken children's interest in measurement and may lead to a negative attitude toward it.

The remaining sections of this chapter focus on the metric system except for the parts dealing with time and money. Teachers who want to introduce customary units first can use the information from these sections as a guide for planning activities by substituting customary units for their metric counterparts.

The progression from working with nonstandard units to standard units goes smoothly for children experienced in working at learning centers or in a laboratory setting. Instead of pencil or cutout-foot units for linear measure, they now use centimeter and meter rulers. Sets of kilogram weights replace walnuts and blocks, and liter containers replace cans.

Linear Measures

The commonly used units of linear measure are the meter, centimeter, millimeter, and kilometer. By the time children complete the sixth grade, they should demonstrate their knowledge of these units by being able to show a reasonable approximation of each and use devices for measuring meters, centimeters, and millimeters accurately. They should also be able to name the other units of linear measure—decimeter, dekameter, and hectometer—and identify the relationships between these and the more common units.

FIGURE 13-3

Ruler for use by youngsters. Only unit segments are marked.

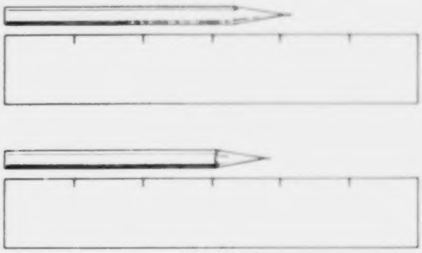
Children will generally begin work with the centimeter. When introduced to this unit, children should use rulers marked only in units; that is, the units should not be marked with numerals (Figure 13-3). Children are forced to count units as they use these rulers; thus their understanding of the meaning of linear measure is reinforced as they work. If such rulers cannot be bought, they can be easily duplicated on tagboard.

Objects whose lengths are a whole number of centimeters should be measured first. This can pose a problem because many familiar objects do not possess this dimension. You can locate appropriate articles for children to measure by looking around the classroom and your home. For instance, a role of 35-millimeter film is packaged in a box that is 4 centimeters wide and 6 centimeters long. A Parker ballpoint pen is 1 centimeter thick and 13 centimeters long. You can also cut sticks, sharpen pencils, and trim paper edges so they have centimeter dimensions.

As children's activities continue, you will want them to measure things that have not been preselected. As they do, they will encounter objects whose dimensions are not a whole number of centimeters. Children will note that 1 pencil is a little more than 4 centimeters long, and another is a little less than 4 centimeters long; Figure 13-4 is an example of this. Even so, each pencil is 4 centimeters to the nearest whole unit.

FIGURE 13-4

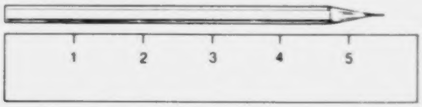
Unit rulers are used to measure the length of two pencils. Each pencil has a measurement of 4 units.



When the end of an object falls near the midpoint between 2 units, children must make a judgment about the measurement to be recorded as its length (Figure 13-5). It should be clear that different individuals may record

FIGURE 13-5

This pencil might be recorded as having a measurement of 5 units by some children and 6 units by others.



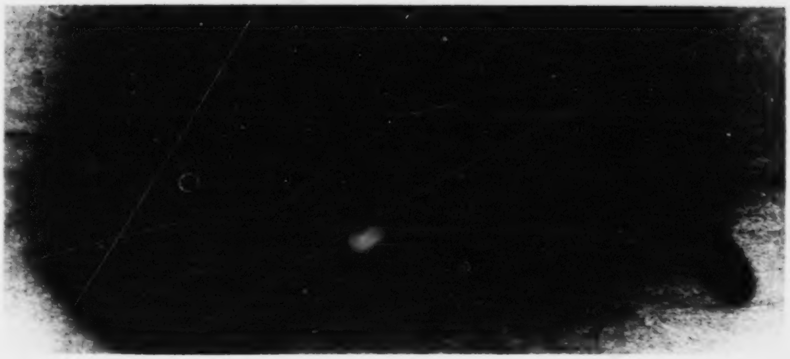


FIGURE 13-6

A metric trundle wheel is rolled across the ground to measure distance.

either of 2 different measurements for this pencil. The measurement might be recorded as 5 centimeters by 1 child and 6 centimeters by another. Both answers are considered correct.

The ruler in Figure 13-5 has units indicated by numerals. Now a measure can be determined by noting the numeral rather than counting the units.

Work with the meter follows the centimeter. The first meter stick should be one that has no subdivisions. Inexpensive pine molding from a lumber yard cut into meter lengths is a convenient source of such sticks. Have children measure the width, length, and height of their classroom, the distance from the classroom door to the cafeteria door, the length of the sidewalk from curb to front door, and so on.

A wheel having a circumference of 1 meter attached to a handle so it rolls easily across flat surfaces is handy for measuring longer distances. Such a device is called a *trundle wheel* and can be purchased or the parts cut from plywood by a teacher and assembled by children (Figure 13-6).

Eventually children should work with meter sticks subdivided into decimeters and centimeters. Metric measuring tapes and millimeter rulers should also be available. Now children will measure all sorts of objects, including themselves. Measurements will be recorded in different ways depending upon children's maturity. For instance, a child's height might be recorded as 148 cm or 1.48 m. Mixed units are not used to record measurements. Thus the above height measurement would not be recorded as 1 m 48 cm.

As their understanding of the metric system matures, children should be guided toward recognition of the fact that a given measurement can be converted from 1 unit to another by shifting the decimal point in 1 direction or the other. Moving the decimal point 1 position to the left is the same as dividing by 10; moving it 2 positions to the left is the same as dividing by 100; and so on. Moving the decimal point to the right 1 position is the same as multiplying by 10; 2 positions is the same as multiplying by 100; and so on. To change 148 centimeters to meters, divide 148 by 100 (there are 100 centimeters in 1 meter). The answer is 1.48; hence, 148 cm = 1.48 m. To change 3.26 meters to millimeters, multiply 3.26 by 1000. The answer is 3260, so 3.26 m = 3260 mm.

One objective of activities dealing with linear measure is that children be able to show reasonable approximations of the meter, decimeter, millimeter, and kilometer. One way to help them do this is to give them opportunities to estimate lengths and distances as well as measure them. Identify a list of objects to be measured. Have each child first record an estimate of the length of each object; then have the child measure the objects and compare the 2 sets of figures.

Another way to meet this objective is to help children establish references based on familiar objects. A dime is approximately 1 millimeter thick. The width of the nail on a person's little finger is about 1 centimeter. The chalk rail in many classrooms is about 1 meter above the floor. Each child can establish a personal standard for measuring meters. Activities 13-9 and 13-10 show how.

ACTIVITY 13-9 Your Step and Meters

Measure a distance of 10 meters in the classroom or a hallway; mark a beginning and ending point.

Have each child walk from start to finish, counting each step; do this three times, recording the number of steps each time.

Have each child determine the average number of steps required to walk 10 meters.

Help each child to remember his or her individual number to use as a reference for activities like Activity 13-10.

ACTIVITY 13-10 How Much Is 50 Meters?

Take the children to a playground and align them along a line marked on the hardtop (or an edge of a sidewalk).

Instruct the children to wait for your signal, then to walk across the playground until each one thinks he or she has walked 50 meters, when each one is to stop.

Have a pair of children use a metric trundle wheel to measure 50 meters.

- Evaluate the children's estimates.
- Have children begin at the start line and estimate other distances, such as 20 meters, 40 meters, and 100 meters. Evaluate estimates each time.

The best way for children to gain a "feeling" for the kilometer is for them to walk the distance several times. Locate landmarks that are about a kilometer from a given point at school—a child's house, a store, a playground, a bridge. As children walk from the point at school to a point at each landmark, have them count their paces. By doing this several times, the children can determine the average number of paces they take while walking a kilometer.

Weight (Mass) Measures

Technically, weight and mass are not the same. Weight is the force exerted on an object because of gravity, whereas mass is the quantity of matter of which the object is composed. Thus a given object will weigh less on the moon, where the force of gravity is less, than it will weigh on Earth. It will have the same mass in either place, because the quantity of matter remains the same. Even though this technical difference exists, it is common practice to consider weight and mass to be the same. Therefore the term that is commonly used—weight—will be used in the rest of this chapter.

The basic unit of weight in the metric system is the kilogram. As pointed out earlier, this unit contradicts the notion that basic units have no prefixes. The commonly used units are

1 metric ton = 1000 kilograms
1 kilogram = 1000 grams
1 gram = 1000 milligrams

By the time children complete the elementary school, they should be able to name these as commonly used units and identify familiar objects that weigh about 1 kilogram or 1 gram. They should also be able to name the less commonly used units, that is, hectogram, dekagram, decigram, and centigram, and state the relationships between each unit and those that are larger or smaller.

Two types of scales should now be available, balance scales and spring scales. In order to weigh objects on a balance scale, a set of weights is needed. A commercial set may contain these weights: one 1-kilogram, two 500-gram, two 250-gram, two 100-gram, two 50-gram, four 10-gram, two 5-gram, and five 1-gram. You can duplicate most of these by filling cloth bags with the appropriate amount of sand or rice. Lead fish weights can be used for the smaller ones. Kitchen-type spring scales that can weigh to 5 kilograms, and platform (bathroom) spring scales are both needed.

ACTIVITY 13-11

Weighing Produce

Provide a balance scale, set of metric weights, and bags of fresh produce commonly sold by weight.

Give each child a work sheet that lists the items and provides spaces for recording their weights.

Children weigh the bagged produce and record their weights, using grams as the unit of measure.

Children should also have opportunities to weigh the same or similar items using a spring scale.

Capacity Measures

The liter is the basic unit of capacity measure in the metric system. It is a derived unit based on a part of the meter—the centimeter. A cube with inside dimensions of 1 decimeter has a volume of 1 cubic decimeter (dm^3), or 1000 cubic centimeters (cm^3); this is 1 liter. Any container having a volume of 1 cubic decimeter has a capacity of 1 liter. Plastic containers of various sizes are inexpensive and can be purchased for classroom investigations. The metric containers in which soft drinks, wine, medicines, and similar products are sold should not be overlooked. By choosing carefully, you can accumulate a collection of graduated containers for your children.

ACTIVITY 13-12

Liters

Set up a center that has a liter container and other containers, such as soft drink, detergent, and vinegar bottles. Have some bottles that hold less and some that hold more than a liter. Label one container *A*, another *B*, and so on; do not label the liter container.

Provide a problem card and answer sheet:

- ☐ Guess which containers hold less than a liter; list their letters under the heading *Less than a liter*. Guess which containers hold more than a liter and list their letters under *More than a liter*.

- ☐ Fill the liter container with water. Fill the *A* container with water from the liter container. Was your guess about the size of that container a correct one? If your guess was not correct, mark out the letter *A* on your answer sheet and put it under the correct heading.
- ☐ Fill each of the other containers from the liter container to determine whether or not your guess for it is a correct one. Mark out any letter that is under the wrong heading and write it under the correct heading.

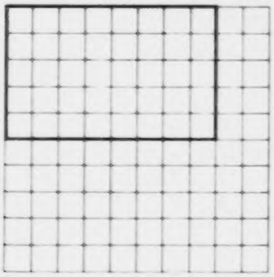
Other centers can provide opportunities for work with milliliters. Specially prepared milliliter containers are available from school supply houses. Small plastic medicine containers cost little or nothing. Children can use these containers to determine the capacity of small bottles, such as those that hold spices and food colors.

Area Measure

The units of measure for the area of the interior region bounded by a closed plane figure are derived from linear units. The metric units that elementary-school children learn are derived from the centimeter and the meter. A square unit with 1-centimeter sides is used for determining the area of small plane figures. It is a centimeter square and is designated by the symbol cm^2. A square with 1-meter sides is used for determining areas of larger regions. Its symbol is m^2. Goals of instruction are for childen (a) to be able to name these common units and (b) to be able to measure area directly by using square centimeter grids and indirectly by first measuring the edges of regions with centimeter and meter rulers and then applying the formula $l \times w = A$.

ACTIVITY 13-13 Finding the Area

Give each child a piece of tissue or plastic that has a grid of centimeter squares marked on it and a page with outlines of rectangular regions on it. Use regions that have a whole number of centimeters along each side. Show the children how to align the tissue or plastic grid on top of a rectangle so that the number of square units contained in its interior can be counted.



Children will measure each of the regions and record its measure.

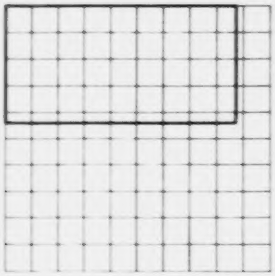
Squares of corrugated cardboard cut with 1-meter edges can be used to measure larger regions. Children can take their squares to the playground or multipurpose room to cover and measure the interiors of game-playing regions bounded by lines painted on the hardtop or floor.

Not all regions can be measured in an exact number of square centimeter or meter units. A square centimeter grid can be used to give an estimate of the measure of such a region's interior. The situation is illustrated in Figure 13-7. First, children should count the units entirely within the bounds of the region—there are 32; then they should count those entirely or partially within the region's bounds—there are 45. The region's measure will be greater than 32 but less than 45; a good estimate is 38.

The next experiences should enable children to state the general formula for finding the measure of any rectangular region using square units: $A = l \times w$ (A represents the measure of the area, l and w represent the measure of length and width, respectively). As they use their grids to measure regions, children should be guided to note that each time they measure a region by counting, they are counting the units in an array. By now the children should know that the product of an a-by-b array is the product of a times b. If a 1-centimeter unit is used to measure the length and width of a rectangu-

FIGURE 13-7

Grid used to estimate a region's area



lar region, the two measures indicate the array of square units for that region. The picture for Activity 13-13 shows a 5-by-8 array; the measurement in square centimeters is 40.

Once children understand how to find the area of rectangular regions, they should determine formulas for the areas of regions bounded by parallelograms.

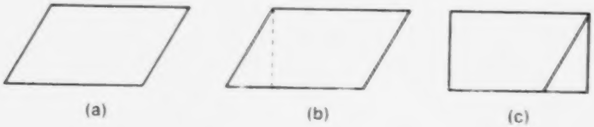
ACTIVITY 13-14

Area of a Parallelogram

Give each child a piece of paper cut in the shape of a parallelogram similar to the one shown in (a).

Instruct the children to fold 1 end of the parallelogram as in (b).

Show the children how to cut off the folded end and put it at the other end to make a rectangle, as in (c).



Review the formula for determining the area of a rectangle, and point out that it can be used to determine the area of a parallelogram. (The formula for a parallelogram is sometimes stated as $b \times h = A$, where b represents base, h height, and A area.)

Children also should explore with triangular-shaped regions. Two congruent triangular regions can be placed side by side to form a parallelogram-shaped region (Figure 13-8). The measure of this area is determined by the formula $A = b \times h$. Because the area for only 1 triangular region is to be determined, the product of $b \times h$ must be divided by 2; this can be expressed by the formula

$$A = \frac{b \times h}{2}$$

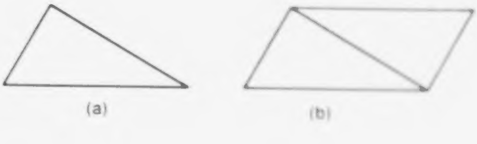


FIGURE 13-8

Two congruent triangular regions can form the shape of a parallelogram, a fact that can help children derive the formula for determining the area of a triangle.

Perimeter Measure

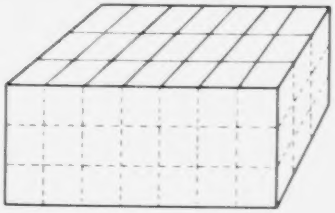
The perimeter of a plane figure is determined by finding the measure of each of the figure's segments and adding the numbers. Children frequently confuse perimeter with area and use a formula for determining area when working with perimeter or vice versa. One way to prevent this confusion is to relate the word *perimeter* to the word *periscope*. Nearly all fifth- and sixth-grade children know that a submarine's periscope is used to look around the surface of a body of water. Point out that the prefix *peri-* means *around*, so perimeter means "to measure around." Once the meaning is clear, children can use centimeter rulers to determine the perimeters of book covers, desk tops, and other small regions. The perimeters of classrooms, playing areas marked on the playground, and larger regions can be determined by using meter tapes or a metric trundle wheel.

Volume Measure

Fifth- and sixth-grade children should learn about the measure of closed three-dimensional figures by making direct measurements of small containers with Cuisenaire cubes that have edges 1 cubic centimeter long (Figure 13-9). Each cube has a volume of 1 cubic centimeter; the symbol for this measure is cm^3. Children should also see a model of a cubic meter. They can make one of corrugated cardboard cut from large boxes.

FIGURE 13-9

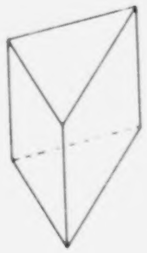
A rectangular solid (box) filled with cubes helps children learn the formula for finding the volume of the solid.



Their experiences with centimeter cubes should enable them to see that indirect measurement can be used to determine the volume of a solid. This is done by determining the measure of the length, width, and height of a solid and multiplying the 3 numbers. The formula that is usually associated with volume is $l \times w \times h = V$. Children who have learned to determine areas of rectangles and triangles can use that knowledge to apply the formula $V = B \times h$, where B designates the area of the base, to determine volumes of rectangular and triangular prisms (Figure 13-10).

FIGURE 13-10

Children can find the volume of a simple geometric solid by using the formula V = B × h.



Measuring Angles

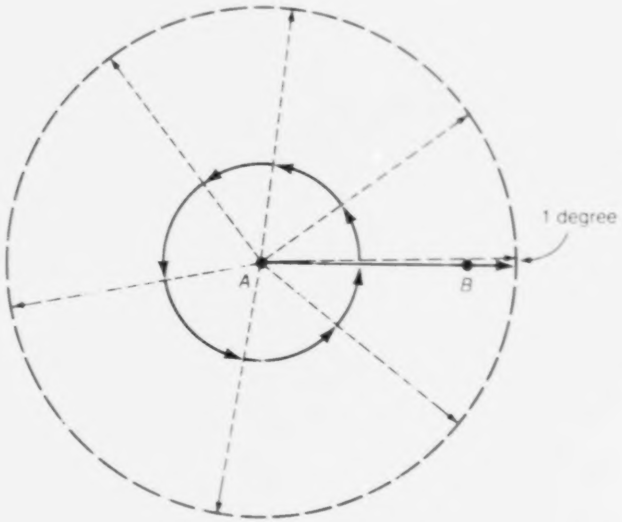
Another type of measure taught in elementary school is the measure of angles. First, the nature of this type of measure must be explored; normally a need to measure angles is not encountered during early years in school.

The measurement of angles introduces a new type of unit—the degree. The meaning of a degree can be explained in this manner: Imagine a ray that is moved around its end point so that it completes 1 full revolution (Figure 13-11). If the ray is stopped 360 times, with all stops being an equal distance apart, the distance from 1 stop to the next has a measure of 1 degree. A protractor, illustrated in Figure 13-13, is used for measuring angles.

Children should not be hurried into using a protractor. They should

FIGURE 13-11

Ray AB *is swept through 1 revolution to illustrate the meaning of 1 degree*



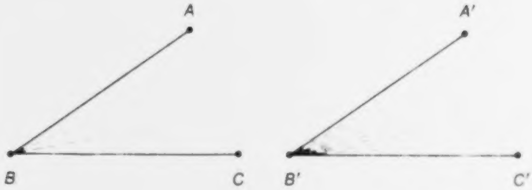


FIGURE 13-12

The measure of the angle ABC *is not as precise as the measure of angle* A'B'C' *because a larger unit of measure is used.*

learn to use one only after they become familiar with angles through the study of nonmetric geometry. They can then begin their study of protractors by making their own, using nonstandard units. They can cut units from cardboard or mark them on clear plastic or tissue paper. While a given child's protractor will have uniform units, a group of children's protractors will have different-sized units. As they use their protractors to measure angles, children will discover that smaller units are more precise and therefore more useful than larger ones (Figure 13-12).

After children understand the basis for measuring angles, introduce the standard protractor by using a large demonstration protractor or a clear plastic one that can be projected on a screen by an overhead projector. As children observe the demonstration or projected protractor, they should discuss the meaning of the marks and numerals on it. Then instruct them in its use by measuring several angles drawn on the board or projected on the screen. Children should realize how important it is to be careful in measuring with a protractor; the point at its center must be on the vertex of the angle, and the base of the protractor must be placed along 1 ray of the angle (Figure 13-13). Each child should then be given a protractor with which to measure angles. Each measurement should be recorded to the nearest whole degree. The typical protractor that children use does not permit measurements any more precise than this.

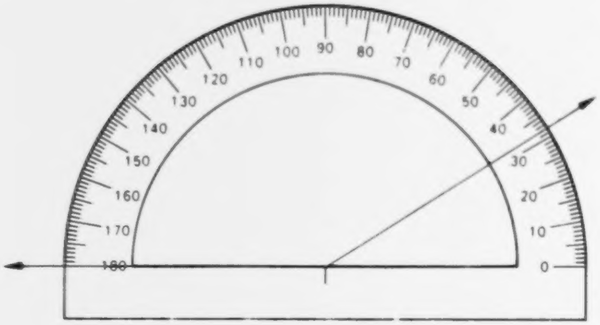


FIGURE 13-13

The correct alignment of a protractor to measure an angle

MINI-OLYMPICS

A mini-Olympics is an entertaining way to provide practice with metric measuring instruments. The activity can be conducted indoors or outdoors and with 1 or more classes. Establish the events, prepare a large chart to record results, and determine what, if any, prizes will be awarded.

Events can be imitations of real Olympic contests or can be invented just for your mini-Olympics.

ACTIVITY 13-15 Cotton-ball Put

The object is to "put" a cotton ball the farthest.

Each contestant has a cotton ball about the size of a tennis ball. A contestant stands at a line and "puts," that is, tosses the ball like a shot put is tossed at a track meet. The distance for each put is measured to the nearest centimeter.

The winner is the person to put a cottonball the greatest distance.

ACTIVITY 13-16 Soda-straw Javelin

The object is to hurl a soda straw the farthest.

Each contestant has a plastic soda straw. A contestant holds the straw between the thumb and middle finger of the throwing hand. The forefinger is at 1 end of the straw. The arm is extended and swung in an arc to hurl the straw. Measure from the starting line to the straw's nearest end with a centimeter tape.

The winner is the contestant whose straw was hurled the greatest distance.

ACTIVITY 13-17 Paper-plate Discus

The object is to throw a paper plate the farthest.

Each contestant has a paper plate. (Make sure each plate is like all other plates.) A contestant stands at a line and throws the plate with a discus-throwing motion. (An alternate way of throwing is a frisbee-throwing motion.) Measure each throw to the nearest meter.

The winner is the contestant whose plate went the greatest distance.

ACTIVITY 13-18 Standing Broad Jump

- The object is to jump the farthest.
- Each contestant stands with feet together and toes on a line. A contestant moves to a crouching position with hands swung back parallel to the ground, then jumps. Measure the jump from the line to the contestant's nearer heel with a centimeter tape.
- The winner is the contestant with the longest jump.

ACTIVITY 13-19 Standing High Jump

- The object is to make the highest mark on a wall.
- Each contestant holds a piece of chalk in 1 hand and stands with that hand next to a wall. To jump, a contestant crouches down, then jumps as high as possible. The contestant makes a mark at the highest point of the jump. Measure the jump from the floor to the highest part of the mark with a centimeter tape.
- The winner is the contestant who jumps the highest.

ACTIVITY 13-20 Build a Clay Bridge

- The object is to build the longest "bridge."
- Contestants compete in pairs. Each team measures a 100-gram piece of Plasticine clay; a judge verifies the weight.
- A bridge consists of a strand rolled from the clay. It is stretched between 2 chairs, with no more than 10 centimeters of the strand on each chair.
- The winner is the team that makes the longest bridge that remains in place for at least 10 seconds without breaking. Teams have 3 opportunities to construct a bridge. The longest of the 3 bridges is each team's official entry.

ACTIVITY 13-21 Build a Soda-straw Bridge

The object is to build a bridge that will hold the most weight.

Bridges are built in advance of the contest by individuals or teams. These conditions apply:

(a) A bridge may have no more than 50 soda straws that are held together by glue.

(b) A bridge must be at least 75 centimeters long and 20 centimeters wide.

(c) There must be a 10-centimeter-square piece of cardobard at the center.

(d) There is no prescribed form for a bridge.

(e) Two or more straws may not be glued side by side to increase their strength. The straws may be cut into pieces.

On Olympics day, each bridge is placed between 2 chairs, with no more than 5 centimeters of each end on the chair. Weights are stacked on the cardboard square at the center of each bridge.

The winner is the individual or team that constructed the bridge that supported the greatest weight before it collapsed.

THREE-TIER ENRICHMENT PLAN

A plan for enriching the mathematics learning of gifted and talented students is described in Chapter 3. Work with measurement offers opportunities to employ this plan in an intermediate-grade classroom. For the first level of work, you can set up an exploratory center that deals with an occupation that uses specialized measuring instruments. For example, civil engineers use specialized instruments for making land surveys for construction projects. A center can contain pictures and models of transits and other measuring devices used by surveyors. Books about the work of civil engineers; about special projects, such as the Egyptian pyramids or present-day dams, highways, and bridges; and about famous engineers, such as Colonel George W. Goethals, who worked on the Panama Canal, should be at the center. A highway department or construction company engineer can come in to talk about engineering work and explain some of the instruments used. Allow all children to examine materials at the center and to listen to the engineer.

Encourage gifted and talented children, and perhaps others who exhibit a strong interest, to advance to the *second level* of work. These children will spend time reading about the civil engineer's work and famous historical and modern engineering projects. They will also learn to operate an engineer's transit, which is used for measuring angles, determining bearings, and making other construction-site measurements. Some students might do research on the pyramids and make models of measuring instruments used in their construction.

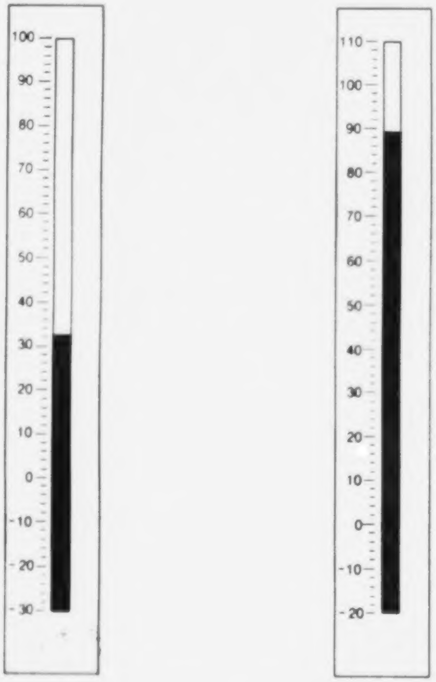
Move these students to the *third level* by giving them responsibility for planning and laying out a baseball diamond, soccer field, or some other game space for the playground or multipurpose room. Students who studied the pyramids can demonstrate the model instruments and share information about the pyramid's construction orally or with a mural or some other means of visual presentation.

MEASURING TEMPERATURE

Children in the second and third grades will learn to use thermometers by reading them and recording temperatures during daily activities connected with weather study. In the higher grades, there are opportunities for chil-

FIGURE 13-14

Demonstration thermometers to help children learn how temperature is determined. The thermometer at left shows metric (Celsius) units, and the one at right shows standard (Fahrenheit) units.



dren to use thermometers as they engage in science studies. By the end of elementary school, children should read temperatures indicated on a Celsius thermometer, know that the freezing point of water is 0°C and the boiling point is 100°C, and identify some temperatures associated closely with their lives—normal body temperature is approximately 37°C, an outdoor temperature of 10°C indicates a cool day, while a temperature of 30°C indicates a hot day (Figure 13–14).

Primary-grade children should use large, easy-to-read alcohol thermometers. You can make a demonstration thermometer by printing the scale on a large piece of stiff cardboard or Masonite painted a light color. The alcohol is represented by a strip of red cloth, made by sewing a strip of red cloth to a strip of white cloth, inserting the ends into slits at the top and bottom of the scale, and sewing the two ends together so the band is taut. After 1 child has read the temperature from the real thermometer, another should show the temperature on the demonstration thermometer so it can be read by the rest of the class.

LEARNING ABOUT TIME

Instruction about time deals with two aspects of the subject: the concept of time, which includes an understanding of the duration of time and the sequence of events; and the mechanics of reading calendars and telling time on a clock. Piaget's research indicates that some children are ready to develop a full understanding of time by the age of nine, and others are not ready until ten or eleven.[8] Before this time, children have a poor sense of duration of time and many misconceptions about the sequence of events. Activities with sand or water flowing through holes in cans and with sand-filled timers, described earlier, are aids for developing a sense of duration. An activity with shadows cast on a simple sundial aids in developing understanding of both the duration and sequencing concepts.

ACTIVITY 13-22 Sundial

Prepare a sundial by attaching a pencil to a square piece of cardboard with Plasticine clay.

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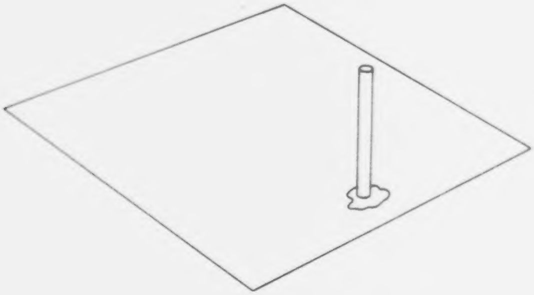
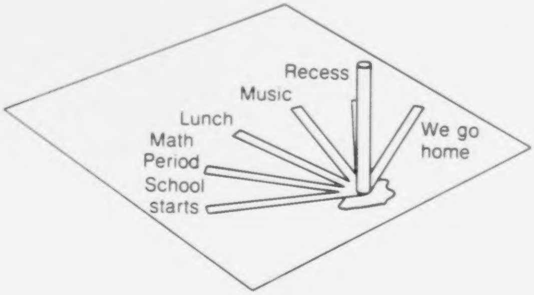
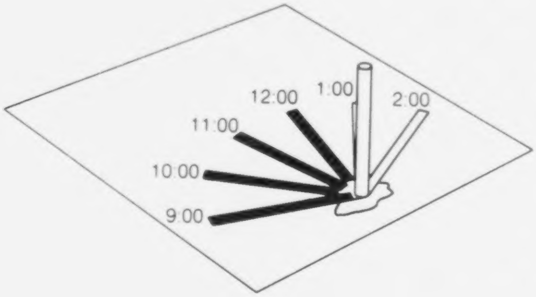
[8] Richard W. Copeland, *How Children Learn Mathematics, Teaching Implications of Piaget's Research*, Third Edition (New York: Macmillan, 1979), p. 209.



- Set the sundial in a south-facing window (or outdoors) when a school day begins. Periodically observe the sundial to note the position of the pencil's shadow. Have a child mark the shadow's position at each observation.
- Note the event that occurs just before or after each observation: "school begins," "math class begins," "recess," "lunch time," and so on.


Recess
Music
Lunch
Math
Period
School
starts
We go
home

- Repeat the activity each day for a week; be certain observations are made at the same time.
- Repeat the activity for a second week but make each observation on-the-hour each day. Mark shadows and the time.


12:00
1:00
2:00
11:00
10:00
9:00

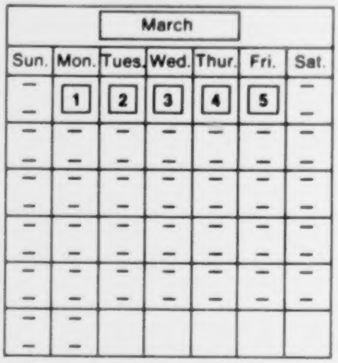
Follow up the sundial activities with discussions that deal with the order of events, and later, with the sequence of hours.

> "This morning we did some work with subtraction, and we made some paper-bag puppets. Which did we do first?"
>
> "We did 3 different things this afternoon. Who can tell me what we did first?" (Listened to a story.) "What did we do next?" (Played a math game.) "What was the last thing we did?" (Saw a film about animals.) "The game came after the story, and the film came after the game. Did the film come before or after the story?" (This latter question helps children see that if event *B* follows *A*, and *C* follows *B*, then *C* follows *A*. This deals with transitivity, a difficult concept for young children to understand.)
>
> "What is the first hour marked on our sundial?" "What is the last hour?" "What hours are in between?" "Let's name the hours in sequence." (9 o'clock, 10 o'clock, 11 o'clock, . . . 2 o'clock.)[9]

Duration and sequence are also features of children's study of the calendar. They learn that calendars indicate days of the week, the number for each day, and the month of the year, as well as the year itself. Many teachers like to incorporate the calendar into seasonal bulletin-board arrangements. For the month of February, birthdays of famous Americans and Valentine's Day can be featured. Calendars such as the one in Figure 13-15, which you can make or buy, can be the basis for daily discussion; as a new date is posted, children should talk about such things as the year, month, day of the week, number of the day, number of days of the month that have already passed, and special days.

FIGURE 13-15

Classroom calendar with removable tabs for recording dates



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[9] This activity is adapted from Charles S. Thompson and John Van de Walle, "A Single-Handed Approach to Telling Time," *Arithmetic Teacher*, XXVIII, No. 8 (November 1981), pp. 4–9.

Children's understanding of time can be enhanced by having them view an entire calendar as 1 continuous number line. Cut up a large calendar and glue or staple the strip for each week in sequence to a long piece of calculator tape. Do not leave a gap between the last day of 1 month and the first day of the next. Affix this new calendar to the walls above chalkboards, with the name of each month centered above its weeks. Labels for each child's birthday and for significant school days and holidays provide a basis for discussion of past and upcoming events. Compare the number of days between Thanksgiving and Christmas and between Christmas and Easter; count the days until the next birthday; and so on.

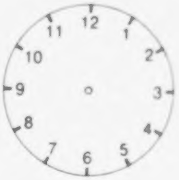
Instruction about telling time should be built around activities with traditional clocks. Telling time with a digital clock should be learned but not at the expense of learning about regular clocks. A clock or watch does more than tell time; it helps us determine how much time has passed since, or is left before, a certain event. A person who says, "A half hour has passed since we began our walk at 3:00" has noted that the minute hand has moved halfway between 3:00 and 4:00. "It is 2:00; we have 45 minutes before the bus leaves at 2:45." A clock face makes it easy to see that 45 minutes remains before the bus leaves.

One recommended sequence of activities for learning to tell time has this order:

> Begin with a clock marked with black hour numerals. Talk about the sequence of numbers. Children should note that 12 is at the top and 6 is at the bottom. The numbers between 12 and 6 and between 6 and 12 should be noted. Prepare a work sheet with 1 large clock face but no numerals. Children write the numerals at each hour mark (Figure 13-16).

FIGURE 13-16

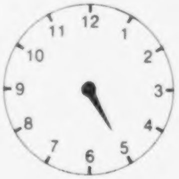
Clock without hands for introducing clocks to children



> Add a black hour hand to the clock. Point it at the 12. Tell children that when the black hand points directly toward a number, it tells the hour. Point it at different numbers and have children name the hour (Figure 13-17).

FIGURE 13-17

Clock face showing hour hand only



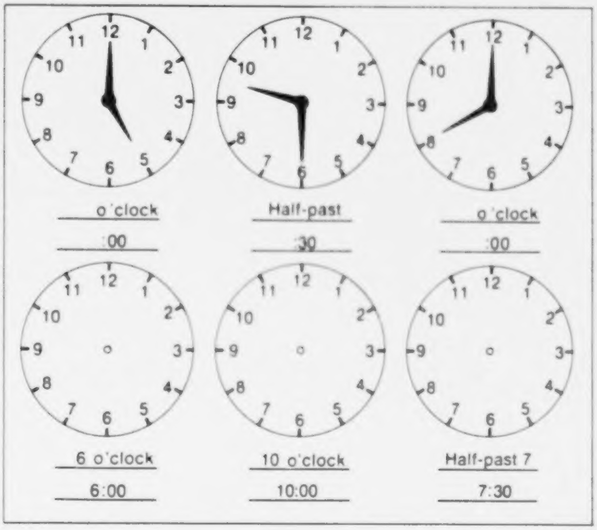
- Show a clock that has black hour numerals and a black hour hand, red minute marks, and a red minute hand. Have children point out differences between this and the first clock. Tell them that when the hour hand points to a numeral and the minute hand points to 12, the time is the hour shown by the hour hand. Set the clock at different times to indicate hours; have children name each hour. Show and discuss two ways of writing time to the hour: 9:00 and 9 o'clock.
- Use a clock with synchronized hands to show the direction the hands move. (Demonstration clocks are available from school supply businesses. Inexpensive mechanical alarm clocks can be used too.) Start with both hands at 12. Have children watch the hands as you turn them. Discuss the fact that while the minute hand makes one complete turn, the hour hand moves from 12 to 1. Repeat, beginning at other hours.
- Practice reading time to the half hour. Emphasize the fact that the minute hand always points to 6 and the hour hand is halfway between two numerals. Set the clock hands at different half-hour positions for children to read.
- Introduce reading time at 5-minute intervals. The ability to count by fives is a useful skill that will make it easier for children to learn to tell time to these intervals. Also point out that 5 minutes pass as the big hand moves from 12 to 1, 1 to 2, and so on. To help children realize this and to grasp the meaning of 5 minutes, have them watch the classroom clock. Remind them often to keep their eyes on the big hand as it moves from 1 numeral to the next as 5 minutes pass. Tell children how to read the clock when the big hand points to a numeral other than 12 or 6, and show them how to write the time, for example, 6:20 and 8:45. To simplify the time-telling process, eliminate expressions like "a quarter past 5," "half past 9," and "a quarter to 10." These times can be expressed as "5:15," "9:30," and "10:45," and read as "five-fifteen," "nine-thirty," and "ten-forty-five," respectively. Time is expressed in figures like these in timetables, TV guides, and other written materials that mention times.
- Introduce times to the minute, using procedures like the ones already mentioned.

Work sheets such as the one illustrated in Figure 13-18 should accompany lessons during which children learn to tell and write times.

Children in the middle grades should continue to practice until they learn to tell time accurately. Some will need to use demonstration clocks and work sheets until they can tell time correctly; others will need no special help. Class activities in the middle grades should also include such topics as the establishment of uniform time zones around the world, the history of telling time, and precision instruments, such as electronic timers, that record time.

FIGURE 13-18

Work sheet for primary-grade children learning to tell time to the hour and half hour



LEARNING ABOUT MONEY

Children receive allowances and spend money at early ages. They need to know the value of each coin and bill in order to use their money properly. Instruction begins in grade one with activities designed to acquaint children with small coins. By the end of grade three, children have experiences with all coins and small bills. Textbooks for these grades often contain realistic pasteboard models of coins. They also contain pages of exercises for which children write the amounts of money represented by pictured coins and indicate amounts of change for simulated purchases.

Even though children work with realistic models and workbook pages, they do not always grasp the relationships between pennies and nickels, nickels and dimes, and so on. These many-to-one and many-to-many matchings are confusing because there is no physical relationship between 1 coin and another to make their values clear. The set of proportional materials described here offers 1 way to show relationships between all coins and a dollar.

> Cut a square with 30-centimeter sides from tagboard. Mark a 10-by-10 grid on it. Label this large square $1.00.
>
> Cut these pieces: 10 small squares, 3 centimeters by 3 centimeters; 5 rectangles, 3 centimeters by 15 centimeters; 5 rectangles, 6 centi-

> meters by 15 centimeters; 4 squares, 15 centimeters by 15 centimeters; and 2 rectangles, 15 centimeters by 30 centimeters. Use cutout pictures of coins or coin stamps to mark the small pieces. A penny goes on each small square, a nickel on each 3-by-15 piece, a dime on each 6-by-15 piece, a quarter on each large square, and a half-dollar on each large rectangle.[10]

Use these proportional models for developing an understanding of each coin's value and of relationships among coins and for change-making activities. Prepare activities that include these types of experiences:

> Cover larger coin pieces with smaller pieces. "How many pennies cover a nickel?" "How many nickels cover a dime?" "What different combinations will cover a quarter?"
>
> Use the pieces to represent different amounts of money. "Show me seven cents." "Show me thirty-two cents." "What are the fewest coins you can use to make sixty-seven cents?"
>
> Introduce numerals for naming the values of different coins: 1¢, 5¢, 10¢, 25¢, and 50¢. (The decimal forms, .01, .05, .10, .25, and .50, are not introduced until values greater than $1.00 are learned.) Show an amount of money, say 53 cents. Have children show that amount with the smallest number of pieces.
>
> Use the $1.00 mat and coin pieces for change-making activities. Display a picture of a small toy and a price. If the toy's price is 35 cents, the children put 35-cents-worth of pieces in the upper left corner of the mat. The amount of change to receive from a dollar is the value of the uncovered portion of the mat. If the purchase is made with a half-dollar, use only the half of the grid in which the coin pieces are located.
>
> Gradually abandon the grid pieces in favor of real coins or models as children show they are ready to change.

Children who are familiar with coins and small bills need to maintain and expand their understandings by regularly engaging in real or contrived activities that require the use of money. These are typical of the activities that are recommended for older children:

> Give each child a real or simulated menu from a favorite fast-food restaurant. An activity sheet with directions such as these can guide children's work:

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[10] This set of materials is described by Stephen J. Ginaitis in "Sense with Cents," *Arithmetic Teacher*, XXV, No. 4 (January 1978), p. 43. Some of the activities that follow are adapted from this article.

- Choose your favorite lunch. List each item and its cost. What is the cost of your favorite lunch?
- If you pay for your favorite lunch with a five-dollar bill, how much change will you get back?
- What combinations of 3 items can you get for less than $1.00?

Plan 7 days of simple menus from foods listed in grocery advertisements. Students must decide on foods and estimate quantities. Use the calculator to sum all the costs. Compare all of the totals to see what the average weekly menu for the entire class costs. Some children may compare this to their family food budget to see if the class plan is reasonable.

Use the calculator and a mail-order catalog or circular to create a shopping spree for students. An imaginary $100 is enough to get students started on preparing a shopping list. Remind intermediate-grade students that tax is paid on purchases. A tax chart or the % key on the calculator can be used to figure tax.

Older children can set up and operate a school supplies business. Pencils, paper, notebooks, erasers, school T-shirts, and so forth can be stocked and sold.

One teacher helped children organize a corporation for which they sold stock to purchase hot-food dispensers for the school's adult lunchroom. The children learned about time purchases, accounting procedures, profits and losses, inventory control, stock dividends, and other aspects of running a business.

SUMMARY OF MEASUREMENT CONCEPTS

As children progress in their study of measurement, they should periodically summarize the concepts they have learned. By the time children have completed the elementary school, most will have mastered these concepts:

1. The process of measuring means selecting a unit of measure and applying it to whatever is being measured. The result gives a comparison between the unit and the item being measured. The number of times the unit can be laid end to end with no overlapping or gaps (in the case of linear measure) is its measure. When the unit of measure is named, for example, 6 centimeters, the item's measurement is stated.
2. Measurements are determined for quantities that are continuous rather than discrete. Because the quantities are continuous, all measurements are *approximate* rather than exact.

3. Units of measure must be of the same nature as what is being measured. For example, line segments are units used for determining linear measurements, and square regions are units for determining area measurements.
4. Units of measure must be uniform wherever they are used. A government establishes standards that guarantee uniformity throughout the country.
5. A measuring instrument is selected on the basis of the job to be done. Generally the smaller the object to be measured, the more precise the instrument. A meter stick or trundle wheel is suitable for measuring a school yard, whereas a centimeter ruler is used to determine the dimensions of a desk top.

Summary

At the time the Metric Conversion Act was signed by President Ford in 1975, it was anticipated that within a few years the United States would become a metric nation. Plans were established for federal agencies to change to metric measures for highway markers, speed limits, and weather reports. Private-sector responses in the automobile and other industries were positive. However, public acceptance of the plan was poor, and because the law was not mandatory, the strong resistance resulted in virtual abandonment of the plan to change from the customary to the metric system. The U.S. Metric Board was abolished, and change in both governmental and private organizations has become nil. For the foreseeable future, children in schools will need to learn about and become skillful in using both the metric and the customary systems of measures.

Children's first activities should be exploratory; they should pour water, sand, and rice from 1 container to another, feel temperature differences, put objects on balance scales, and order objects by length, height, capacity, and size in order to build background for understanding linear, weight, capacity, area, time, and temperature concepts. Nonstandard units are used to introduce the meanings of different measures—linear, weight, capacity, area, and time. Later, various commonly used metric and customary units are introduced, and children learn the instruments and processes for using each one. Learning-center activities guided by problem-card directions are useful for giving children measuring experiences. By the time children complete the elementary school, they should be able to name commonly used metric and customary units, describe the structure of the metric system, and use measuring instruments accurately. Special units are used to acquaint children with time concepts, to read time from a clock, and to understand and use money effectively.

Study Questions and Activities

1. Many books dealing with the metric system have been written for children. Read at least 5 of these books and write a review of each. Include author, title, publisher, date, a brief review of contents, and the age of children for whom the book is intended. Rate the books on a scale of 1 to 5. What qualities make highest-rated books so good? What problems did you find with the lowest-rated books?
2. Several articles in the For Further Reading section contain games and puzzles for teaching about the metric system, time, and money. Add to your teaching file by finding activities on measurement topics, including linear measure, area measure, weight and capacity measure, time, and money.
3. Create a bingo activity for intermediate students to practice metric measures, abbreviations, prefixes, and values. One bingo activity would be to put the metric abbreviations (m, L, kg, and so on) and the prefixes on the bingo cards. The call sheet would be the metric units (meter, centimeter, liter, gram, kilogram, and so on) and decimal values (1000, 100, 10, 0.1, and so on). Make 2 other sets of clues for the bingo cards. One might include clues such as "1/1000 of a meter" and "100 grams." A second set might emphasize the understanding of the size, with clues such as "the width of a little finger" or "the distance from the flagpole to the red light at the corner" or "the weight of the dictionary." The rules of bingo might be changed for the last set of clues to allow 2 blocks on a card to be covered from 1 clue; for example, 10 and dm for dekameter.
4. A 3-level enrichment plan recommended by Ridge and Renzulli for gifted and talented students is described in Chapter 3. Review that plan and the example described in this chapter. Then design a brief unit that follows their scheme to enrich fifth- or sixth-grade children's knowledge of measurement. You might use an occupation, such as meteorologist, and its specialized measuring instruments as the subject of your unit.
5. Read the article by James Hiebert (see For Further Reading section) about the difficulties students have with measurement. Write down the major problems that he finds. What kinds of experiences are needed to overcome these difficulties? See if you can find activities in this chapter that address each of the problems. Look for additional activities to deal with other problems he lists.
6. In the age of digital clocks, should children still learn how to tell time on the traditional clock? Discuss this question with 2 or 3 friends. What is the consensus, and what reasons are given? Many textbooks teach terms such as quarter past and quarter before the hour. Why do you feel children have trouble with this terminology? Would 2:15 and 2:45 be better phrasing for children in the digital age?

7. John W. Bradford (see For Further Reading section) describes a way to use Cuisenaire rods and plastic grids to help children learn the value of each U.S. coin. Read the article and compare his materials and procedures with the tagboard grid described in this chapter. What are the strengths and weaknesses of each set of materials? Which materials would you prefer to use with first- or second-grade children? Why? Do you think that 1 set of materials would be better for working with older children who have not learned the value of each coin and how to make change? If so, which set and why?
8. Read Marion Walter's article (see For Further Reading section). Use her examples as a basis for designing problem cards dealing with perimeter and area that are suitable for fifth- and sixth-grade children.

For Further Reading

Bradford, John W. "Making Sense Out of Dollars and Cents," *Arithmetic Teacher,* XXVII, No. 7 (March 1980), pp. 44–46. Cuisenaire rods and plastic hundred-grids serve as models for coins and dollar bills. Procedures for moving from the concrete models to a written code for representing money are described.

Carr, Karen D. "A Common Cents Approach to Mathematics," *Arithmetic Teacher,* XXVI, No. 2 (October 1978), pp. 14–15. A sixth-grade class banking project, complete with forms and simulated life experiences, provides a study of the economics of daily living and how to cope with management of checking and savings accounts.

Classon, Robert G. "How Our Decimal Money Began," *Arithmetic Teacher,* XXXIII, No. 5 (January 1986), pp. 30–33. Provides an historical perspective on our money system.

Cooper, Martin. "Leap Years and Such," *Arithmetic Teacher,* XXVII, No. 5 (January 1980), pp. 26–27. Not every fourth year after 1984 is a leap year! Cooper tells why and describes activities about the history of the calendar for children. This work is for interested older children. It offers practical calculator exercises.

Daane, C. J. "Primary Coin Activity Cards," *Arithmetic Teacher,* XXVII, No. 6 (February 1980), pp. 34–36. Coin sheets and a coin trading board are the materials needed for 6 games, with emphasis on pennies, nickels, dimes, and quarters.

Elliott, Portia A. "Money Changer's Bingo," *Arithmetic Teacher,* XXVII, No. 1 (September 1979), pp. 50–51. Money Changer's Bingo is for upper-grade children who need practice in making and counting change. The rules and materials for the game are described.

Ginaitis, Stephen J. "Sense with Cents," *Arithmetic Teacher,* XXV, No. 4 (January 1978), p. 43. Describes procedures using a 10-×-10 hundreds board and smaller regions to represent dollars and cents. Activities with the materials aid children in learning coin values and change-making skills.

Hart, Kathleen. "Which Comes First—Length, Area, or Volume?" *Arithmetic Teacher,* XXXI, No. 9 (May 1984), pp. 16–18, 26. Cognitive development provides clues for teaching about measurement concepts and activities.

Hiebert, James. "Why Do Some Children Have Trouble Learning Measurement Concepts?" *Arithmetic Teacher*, XXXI, No. 7 (March 1984), pp. 19–24. Children's misunderstandings about measurement provide information for teachers about the kinds of activities needed.

Hildreth, David J. "The Use of Strategies in Estimating Measurements," *Arithmetic Teacher*, XXX, No. 5 (January 1983), pp. 50–54. Systematic approach to estimation of measurements increases skill with process of measurement.

Horak, Virginia M., and Willis J. Horak. "Teaching Time with Slit Clocks," *Arithmetic Teacher*, XXX, No. 5 (January 1983), pp. 8–12. Suggests activities for time concepts and telling time with student-made slit clocks.

Lindquist, Mary M., and Marcia E. Dana. "The Neglected Decimeter," *Arithmetic Teacher*, XXV, No. 1 (October 1977), pp. 10–17. The decimeter is more suitable than either the centimeter or meter for introducing children to standard units of linear measure, claim these writers. Five types of activities with many examples of each are presented.

Litwiller, Bonnie H., and David R. Duncan. "Areas of Polygons on Isometric Dot Paper: Pick's Formula Revisited," *Arithmetic Teacher*, XXX, No. 8 (April 1983), pp. 38–40. Suggests how dot paper can be used to calculate areas of irregular polygons.

Nelson, Glenn. "Teaching Time-Telling," *Arithmetic Teacher*, XXIX, No. 9 (May 1982), pp. 31–34. The author's approach to teaching time is one that simplifies the process by eliminating many of the confusing terms that inhibit learning. His simplified materials and terminology, along with a sequence for teaching time-telling, are illustrated and discussed.

Riley, James E. "It's About Time," *Arithmetic Teacher*, XXVIII, No. 2 (October 1980), pp. 12–14. Activities dealing with duration and sequence of time are described.

Steffe, Leslie P. "Thinking about Measurement," *Arithmetic Teacher*, XVIII, No. 5 (May 1971), pp. 332–338. Piaget's theories of cognitive development and some of their applications to learning the meaning of measurement are discussed in this thought-provoking article.

Szetela, Walter, and Douglas T. Owens. "Finding the Area of a Circle: Use a Cake Pan and Leave Out the Pi," *Arithmetic Teacher*, XXXIII, No. 9 (May 1986), pp. 12–18. Presents 7 ways of finding the area of a circle.

Thompson, Charles S., and John Van de Walle. "A Single-Handed Approach to Telling Time," *Arithmetic Teacher*, XXVIII, No. 8 (April 1981), pp. 4–9. Sundials and one-handed clocks provide the basis for activities to introduce children to time concepts and how to read clocks.

Walter, Marion. "A Common Misconception about Area," *Arithmetic Teacher*, XVII, No. 4 (April 1970), pp. 286–289. The common misconception is that figures with common perimeters will have the same area regardless of the measures of their sides. The examples given here can serve as the basis for several problem cards to challenge mathematically mature fifth and sixth graders.

Zalewski, Donald. "Some Dos and Don'ts for Teaching the Metric System," *Arithmetic Teacher*, XXVI, No. 4 (December 1978), p. 17. Briefly discusses a list of 6 "dos" and 2 "don'ts" to consider while teaching the metric system.

Chapter

14

Tables, Graphs, and Probability

In Chapter 14, you will read about

1. sources of data for children's tables and graphs
2. five types of graphs children can know and use by the end of grade six
3. reasons why probability activities should be part of the elementary mathematics curriculum
4. probability activities for both primary and intermediate grades
5. probability terms that fifth- and sixth-grade children can comprehend and use

One of the important problem-solving skills that was discussed in Chapter 5 is the compilation of data and construction of tables and graphs. Not only is it important that students acquire skill in making tables and graphs from their own data, they also need to know how to read and interpret tables and graphs prepared by other persons. That many students cannot use tables and graphs effectively is supported by data from the second *National Assessment of Educational Progress*. Children performed well when they answered questions based on a direct reading of graphs. They did less well when they were required to make predictions or solve problems based on information contained in graphs.[1]

None of the materials needed to help children learn to prepare tables and graphs is expensive or hard to get. Sheets of centimeter-square paper, rolls of plain paper, and paper with inch and 2-centimeter squares, as well as colored cubes, picture stickers, pens and crayons, compasses, protractors, and similar items provide raw materials for the graphs that elementary school children make.

SOURCES OF DATA FOR TABLES AND GRAPHS

Children themselves provide a rich and steady source of data for making tables and graphs. In kindergarten and primary grades, children's birthdays, shoe sizes, means of getting to school, weight, height, hair and eye colors, weekend TV viewing, and favorite sandwich meats are suitable topics. Some first- and second-grade teachers maintain a chart on which the dates children lose teeth are recorded, as well as a graph to show each month's total. By year's end, the totals for each month and the school year have been recorded. The graph then serves as a basis for discussion about the number of teeth lost during the year, the month showing greatest loss, the month showing the least loss, and so on.

Intermediate-grade children are also sources of data about themselves. Topics may deal with their favorite things or activities: TV shows, school subjects, sports, desserts. These children are also capable of preparing and using graphs in science, social studies, and other curriculum areas. During plant growth experiments, a daily or weekly record of different plants' growth must be noted and recorded. In social studies, children can prepare circle graphs to show the distribution of a nation's exports according to product categories; and bar graphs to indicate the wheat tonnage by state for a given year.

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[1] Barbara J. Bestgen, "Making and Interpreting Graphs and Tables: Results and Implications from National Assessment," *Arithmetic Teacher*, XXVIII, No. 4 (December 1980), p. 26.

Data for tables and graphs should be compiled by children themselves whenever possible. Even when the data is not theirs, such as wheat tonnage by state, the need for it should arise from an ongoing activity during which they search for information in encyclopedias, almanacs, and other sources. The many opportunities that are available for obtaining information from the children's environment make virtually unnecessary the use of already prepared material from workbooks, textbooks, and similar sources.

TYPES OF GRAPHS

Kindergarten and primary-grade children's first graphs should be simple. Their first one, in fact, may not resemble a graph at all. Rather the first one may look like stacks of colored blocks arranged on a table's edge. As time goes by, pictures can replace the blocks. Over time, primary-grade children should learn to use simple bar graphs. Children in intermediate grades should learn about picture, bar, line, and circle graphs.

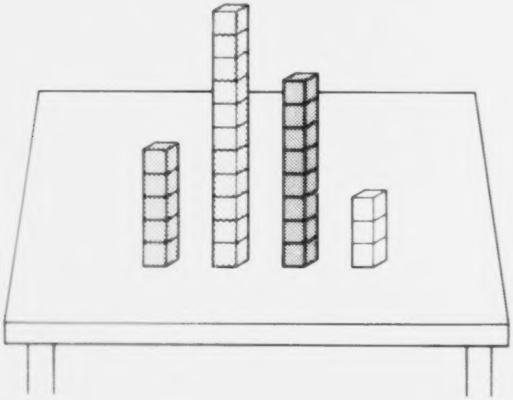
Object Graphs

An object graph is ideal for kindergarten and first-grade children because neither numerals nor words are required. In Activity 14-1, colored blocks are used; in Activity 14-2, plastic animals are used.

ACTIVITY 14-1 The Shoes We Wore Today

Have each child look at and note the color of shoes he/she is wearing. Ask each child wearing brown shoes to raise a hand; give each child a brown wooden cube. (Unifix cubes can be used instead of wooden cubes.) Give each of the other children a cube that matches his/her shoe color.

Have each child who has a brown cube place his/her cube in a stack at the edge of a table in clear view of all children. Have each of the other children do the same. The stacks should be placed side by side.



Ask questions such as: "What color of shoe was worn by the most children today?" "What color was worn by the fewest children?" "Are there any shoe colors of which there are the same number?" "Is there a shoe color worn by just 2 people?"

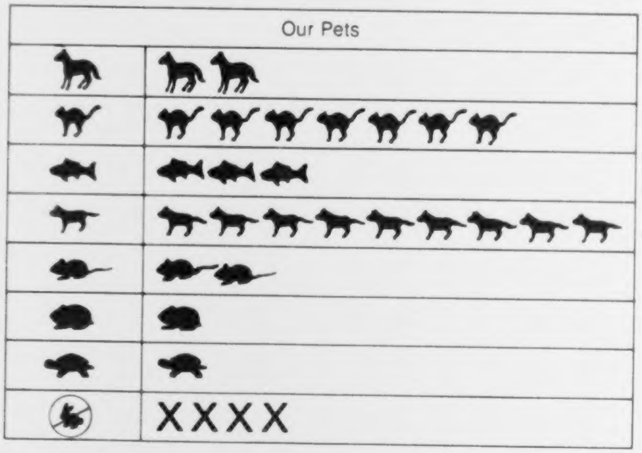
Children's pets are the subject of the next activity.

ACTIVITY 14-2 Pet Statistics

Poll the children about their pets. (You may choose to have children who have more than 1 pet limit their contribution to 1 pet only.)
Tally the results in a simple table.

Our Pets	
Horse	//
Cat	𝍸 //
Fish	///
Dog	𝍸 ////
Mouse	//
Guinea pig	/
Turtle	/
No pet	////

- Give each child a plastic dog, cat, fish, or other pet. Children without pets can be given a small plastic basket to represent an empty pet bed.
- Place a model of each chosen pet on an already prepared graph form.
- Have the children place their plastic animals or baskets on the graph.



- Use the graph for counting and comparing activities:
  - "How many children own cats? dogs? fish?"
  - "Do more children own cats than dogs? dogs than horses?"
  - "How many fewer cat owners are there than dog owners?"
  - "Are there more children who have pets, or more who do not have pets?"

Picture Graphs

The change from object graphs to picture graphs is easily made. The pet survey in Activity 14-2 can easily be recorded on a picture graph. Instead of giving each child a plastic animal, give each one a colorful sticker that shows his or her pet. Be sure that the stickers are all the same size. A line marked on a piece of butcher paper can provide a base line on which each column of animals sits. Caution children to use care when they put stickers on the graph so that they do not have gaps or overlaps. The kinds of questions asked in Activity 14-2 apply here.

Intermediate-grade children should work with more complex picture graphs. In the pet situation, it was not unreasonable to have a picture for each pet. But as the size of the sample from which data is gathered increases, it is less practical to use 1 picture for 1 subject. Problem 14-1 suggests that a picture graph be used to show the results of a school poll about children's favorite TV programs. On this graph, each picture represents 10 students.

PROBLEM 14-1 Favorite TV Shows at Our School

Plan and organize with your children a procedure to poll their schoolmates in kindergarten through grade six to determine each child's favorite TV program.

Assist the children as they prepare a table to record the results of the poll.

Sesame Street	𝍸 𝍸 𝍸 𝍸 𝍸 𝍸
Scooby-Doo	|||
Highway to Heaven	𝍸 𝍸
Wonderful World of Disney	𝍸 𝍸 𝍸 |||
⋮	⋮
Murder, She Wrote	||

Discuss with the children a reasonable way to portray the data in a picture graph. From the information in the table, have 1 picture of a TV screen represent 10 children.

Choose teams of children to enter the data on the graph. Make sure that the children understand how to use a part of the TV-screen picture to represent fewer than 10 children. Use half a picture for 5, more than half for 6, 7, 8, or 9, and less than half for 1, 2, 3, or 4.

Sesame Street	▣ ▣ ▣
Scooby-Doo	◧
Highway to Heaven	▣
Wonderful World of Disney	▣ ◧
⋮	⋮
Murder, She Wrote	◧

▣ = 10 viewers

Discuss the results of the poll: "What TV show is most popular?" "What shows are among the top 5?" "Is any particular type of show—cartoon, adventure, comedy, mystery, educational, and so on—more popular among kindergarten and primary-grade children?" "Among children in the intermediate grade?"

Variation: Use a computer graphing program to display the results of the graph. *Easy Graph*[2] is one graph program designed for children to use.

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[2] Grolier Electronic Publishing, Inc., 95 Madison Ave., New York, NY 10016.

Bar Graphs

Children who have made and interpreted object and picture graphs will find bar graphs easy to use and understand. Data from the same types of sources serve as the source of information for this type of graph. Primary-grade children who have made an object graph of their shoe colors can make a bar graph in a similar way. Now, instead of giving each child a colored block, give a square of colored paper that matches his or her shoe color. The squares of paper are glued to a piece of butcher paper on which a base line has been drawn. Again, caution the children about gaps and overlaps. Older children can maintain a table and graph of shoe colors for a period of time. The table and graph in Figure 14-1 show the results of the first 3 days of a month-long activity. An interesting side activity is to have the children look for any correlation between shoe colors and weather. If a record of the weather is kept (data for another table and graph), the children can match days on the 2 tables and graphs to see if 1 color of shoe was worn more often on a certain type of day than another color. If there appears to be some correlation, see if the children can draw any conclusions and make any predictions based on their findings.

Line Graphs

Line graphs are useful for showing patterns in tables of numbers. The data may come from a variety of sources: colors of shoes worn by children, temperatures for a community, population of a county or state, sales in a department store, average incomes by occupations, and so on. Line graphs are usually introduced in one of the intermediate grades. A fourth-grade class that has been making tables and graphs of shoe colors could easily extend the work to include a line graph portraying the data (Figure 14-2). In order to make a line graph like this, children must understand the significance of each axis. In this example, the horizontal axis indicates the days children are in school during 1 month; the vertical axis indicates the number of shoes worn by the children. More than 1 line occurs on this graph because children wear more than 1 color of shoes. It is necessary to have a key to indicate the line that depicts each shoe color. Help the children note also that the sum of the numbers shown for a given day is the same as the number of children in class for that day.

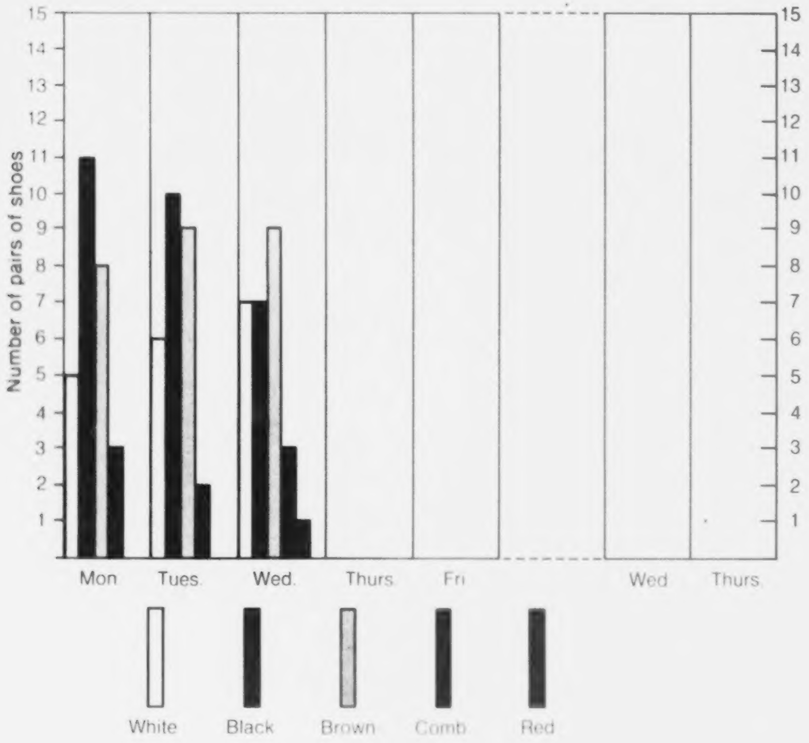
Graphs on a coordinate plane are a form of line graph that is useful for plotting points from tables showing rate pairs, or functions. In Chapter 5, Problem 5-14 illustrates an example of this use of a graph. The problem asked children: How many stars are required for 16 ornaments if 7 stars are used for one ornament? That problem is repeated here to show how you might introduce this use of a line graph to fifth- or sixth-grade children.

FIGURE 14-1

Table (a) and bar graph (b) with data concerning colors of children's shoes

Days Shoe Colors	Mon	Tues	Wed	Thurs	Fri	Mon	Tues	Wed	Thurs	Fri	Mon	Tues	Wed	Thurs	Fri	Mon	Tues	Wed	Thurs
White	5	6	7																
Black	11	10	8																
Brown	8	9	9																
Comb	3	2	3																
Red			1																
Totals	27	27	27																

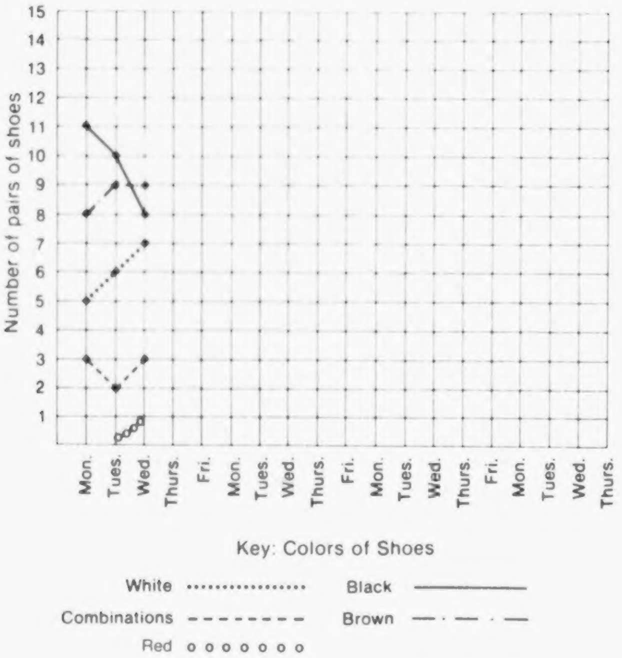
(a)



(b)

FIGURE 14-2

Line graph with data concerning color of children's shoes



PROBLEM 14-2 Making Christmas Ornaments

Present the problem: "Our class is going to make Christmas ornaments to sell. Each ornament will have 7 gold stars. How many stars will be needed for 16 ornaments?

Discuss the problem with the children to determine if any child has an idea about its solution. If a child mentions that this is a rate-pair situation, capitalize on that idea to make a table. If none does, help the children construct a table to show the first 4 ordered pairs.

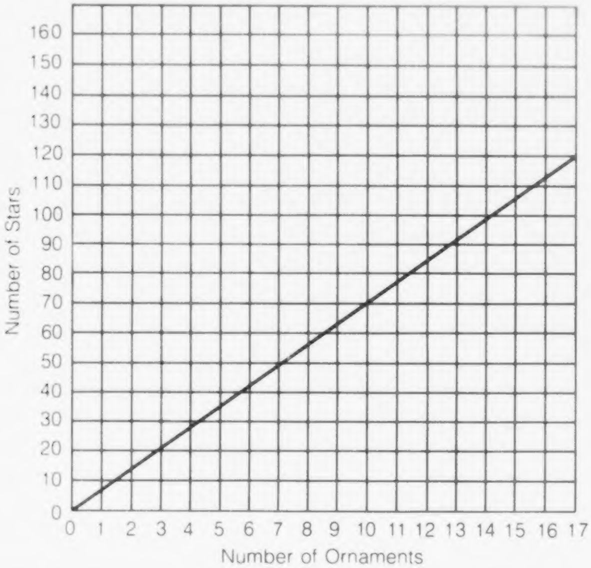
Number of Ornaments	1	2	3	4		15	16	17
Number of Stars	7	14	21	28				

Show the children how to draw a horizontal and a vertical axis on a large piece of 1-inch squared paper. Label the horizontal axis *ornaments*, and number it 1, 2, 3, . . . Point out that the numerals indicate the number of ornaments and that this line is called the "*x* axis." Label the vertical axis *stars* and number it 7, 14, 21, . . . Point out that these numerals indicate the number of stars and that the line is called the "*y* axis."

Have the children compare the first 4 numbers in the table with the numbers along the *x* and *y* axes. Help them see that if they go to the "1" on the horizontal axis, up to the first line, and left to the *y* axis, they are at the number "7." Do the same with 2, 3, and 4.

Next show them how to mark points on the graph to indicate the path on the line for the ordered pairs in the table: 1,7; 2,14; 3,21; 4,28.

Complete the line for this graph.



Show how to determine the number of stars required for 6 ornaments by locating 6 on the *x* axis, reading up to where the line intersects, then looking at the number on the *y* axis. Have children read other ordered pairs from the graph.

Present tables for other ordered pairs for children to graph:

cost	16¢	32¢	48¢	64¢					
number	1	2	3	4					

kilometers	85	170	255	340				
hours	1	2	3	4				

Circle Graphs

Circle graphs are used to portray information about fractional parts of a whole. The whole can be a particular population of people, such as the members of a class, people in a city, or citizens of a country. It can be department store sales for a month or year, or grain production in a state or country for a year or decade. Parts of a circle graph may be represented by common fractions, decimal fractions, or percentages.

Data from sources close to the children can serve as a basis for making circle graphs. Instruction must be planned carefully and presented in an orderly sequence to assure that children grasp the meaning of each step as they prepare this type of graph. Children need a good knowledge of common and decimal fractions and also percent if shares on graphs are recorded in that notation. They must also know how to use a protractor to measure degrees before they can understand and use circle graphs. The following activity uses the TV survey to show how instruction about a circle graph might proceed:

ACTIVITY 14-3

Our TV Survey

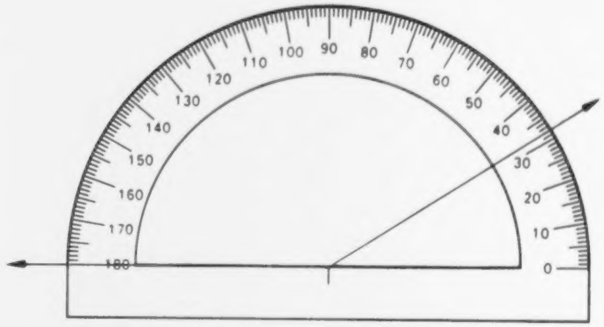
Categorize all the shows according to type—cartoon, adventure, comedy, mystery, educational, and so on.

Cartoon *Adventure* *Comedy* *Mystery* *Educational*

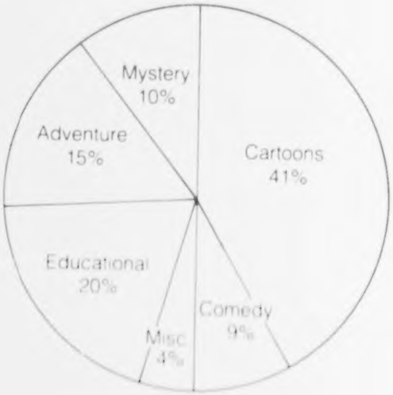
Determine the fractional part of the whole for each particular type of show. For example, if there are 376 children in the school and 153 of them prefer cartoons, the common fraction for that share is 153/376. Point out that even though the fraction could be used, it is not a very effective way to indicate the share for cartoons because its meaning is not as easy to grasp as that of a decimal fraction or percentage.

Children can use a calculator to determine quickly that 0.4069148, or 0.41 (41 percent) is the share for cartoons. Use the same procedure to determine the share for each other type of show.

Help children determine the share of a circle that will represent each decimal fraction or percent. In the example here, help them determine that 0.41 or 41 percent of 360 degrees is approximately 148 degrees. If necessary, show children how to use a protractor to indicate 148 degrees of a circle.



Write the share for each of the other types of shows as a common fraction, and determine the decimal fraction or percentage for each share. Complete the circle graph.



Discuss the meaning of the graph. Point out how to compare 1 part of the graph with another part. The size of 1 part might be about twice the size of another part; the numeral in the larger part will represent a number that is about twice as large as the number represented by the smaller part.

Variation: A simple Logo program can be used to show the pie chart. The PIECHART procedure uses 2 pieces of data to draw each part of the circle.

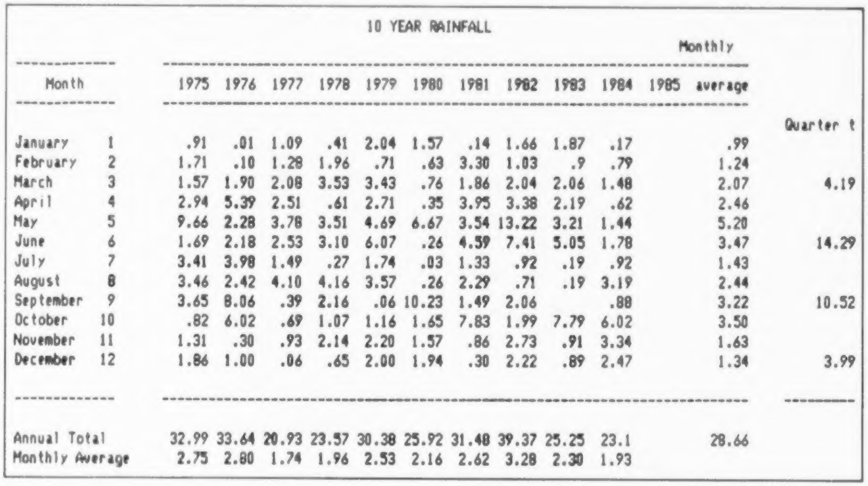
```
TO PIECHART :PART :WHOLE
REPEAT 360* :PART/ :WHOLE [FORWARD 40 BACK 40]
END
```

For the television survey, the part of the pie chart to show cartoons is drawn by typing PIECHART 153 376. Each segment of the pie chart is drawn by entering the number for that part and the total number. Between drawing each segment, change the pen color so that the different parts are clearly shown.

Making charts and graphs is another way for children to use the computer in mathematics. *Exploring Tables and Graphs*[3] is a tutorial program based on games. As children play each game, points are displayed on a bar, line, or circle graph. Then children can work through a lesson that points out how graphs are constructed and make a graph. Another graphing program is *Graphworks*.[4] *Graphworks* takes information from the spreadsheet in *Appleworks*™[5] and creates bar, line, and pie charts with the information. In Chapter 5, a classroom problem was suggested in which students investigate the rainfall for their region. Plotting information about rainfall is a good way to show the patterns and discrepancies of the information. Figure 14-3 contains a chart showing rainfall information obtained from the National Weather Service. The spread sheet allows students to calculate totals and averages for the months and years very easily. Then that information is used in a graph drawn with *Graphworks*. The information can be displayed in several different forms (Figure 14-4).

FIGURE 14-3

Ten-year rainfall chart



```
                                       10 YEAR RAINFALL
                                                                            Monthly
-------------      ---------------------------------------------------------------------
    Month          1975  1976  1977  1978  1979  1980  1981  1982  1983  1984  1985  average
-------------      ---------------------------------------------------------------------
                                                                                                 Quarter t
January     1       .91   .01  1.09   .41  2.04  1.57   .14  1.66  1.87   .17              .99
February    2      1.71   .10  1.28  1.96   .71   .63  3.30  1.03    .9   .79             1.24
March       3      1.57  1.90  2.08  3.53  3.43   .76  1.86  2.04  2.06  1.48             2.07        4.19
April       4      2.94  5.39  2.51   .61  2.71   .35  3.95  3.38  2.19   .62             2.46
May         5      9.66  2.28  3.78  3.51  4.69  6.67  3.54 13.22  3.21  1.44             5.20
June        6      1.69  2.18  2.53  3.10  6.07   .26  4.59  7.41  5.05  1.78             3.47       14.29
July        7      3.41  3.98  1.49   .27  1.74   .03  1.33   .92   .19   .92             1.43
August      8      3.46  2.42  4.10  4.16  3.57   .26  2.29   .71   .19  3.19             2.44
September   9      3.65  8.06   .39  2.16   .06 10.23  1.49  2.06          .88             3.22       10.52
October    10       .82  6.02   .69  1.07  1.16  1.65  7.83  1.99  7.79  6.02             3.50
November   11      1.31   .30   .93  2.14  2.20  1.57   .86  2.73   .91  3.34             1.63
December   12      1.86  1.00   .06   .65  2.00  1.94   .30  2.22   .89  2.47             1.34        3.99

-------------      ---------------------------------------------------------------------      ----------

Annual Total      32.99 33.64 20.93 23.57 30.38 25.92 31.48 39.37 25.25  23.1            28.66
Monthly Average    2.75  2.80  1.74  1.96  2.53  2.16  2.62  3.28  2.30  1.93
```

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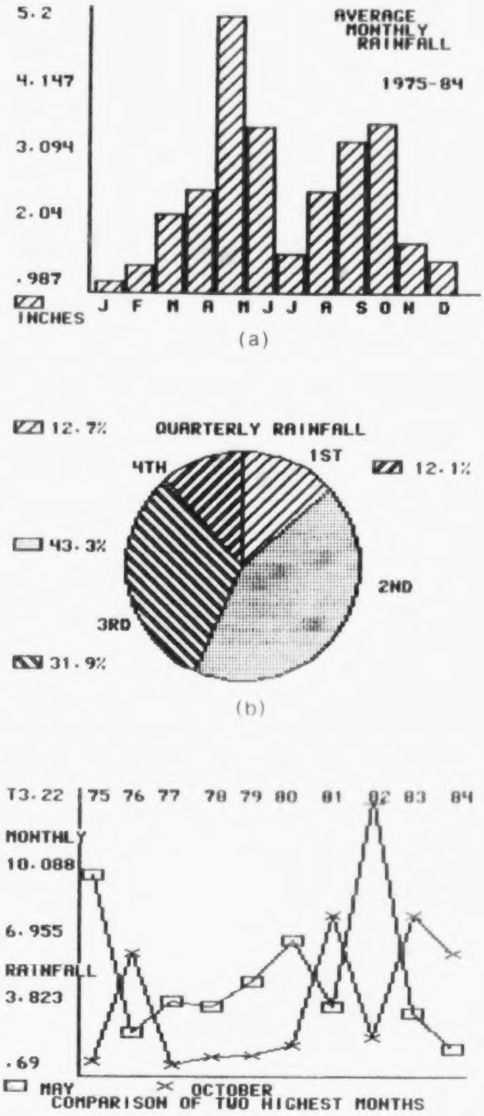
[3] Xerox Educational Publications, 245 Long Hill Road, Middletown CT 06457

[4] PBI Software, 1155B-H Chess Drive, Foster City, CA 94404

[5] Apple Computer Inc., 20525 Mariana Avenue, Cupertino, CA 95014



(a)

(b)

(c)

FIGURE 14-4

The program Graphworks *can prepare a bar, circle, and line graph of a locality's monthly rainfall totals.*

PROBABILITY

Probability deals with chances. What are the chances of an event occurring? If I plan a picnic in July, what are the chances of rain spoiling the event? If I plan one in February, what are the chances of rain? If I roll this die (singular of dice), what are the chances of a "6" being the number on top? Information about the probability of rain in July or February is based on statistical data and processes applied to the data. Statistical data gathered over more than 100 years show that the chances of rain in July in the Central Valley of California are practically nil, so a picnic in July is unlikely to be rained out. The same data show rain to be much more likely in February, so a picnic during that month is more likely to be canceled than one in July. Information about the probability of a "6" showing on a die is based on the mathematics of chance. The probability of getting a "6" is exactly 1 in 6 when the die is fair (that is, constructed so that each face has an equal chance of coming up on top).

The following reasons support the inclusion of activities dealing with probability in the elementary school:

1. Probability activities enhance children's problem-solving skills. Children must establish procedures for carrying out experiments, construct tables, search for patterns, perform computations, report the results of their work, and draw conclusions.
2. The activities provide children with useful knowledge. Individuals are constantly confronted with many instances where information is based on either the mathematics of chance or statistical data. Individuals who understand the origins of the information will interpret it and react to it more effectively than those whose understanding is slight.
3. Probability activities are fun. They provide variety in a mathematics program and challenges for children.

Probability Experiments

Some probability experiments are simple enough for children in the primary grades. The background of mathematical knowledge and skills needed to comprehend more advanced experiments limits them to the intermediate grades. But deciding whether children can cope with an activity should be based on your knowledge of their background and interest in mathematics, not an arbitrary placement of that activity to a particular grade level. The activities presented here are samples of those that are appropriate for helping elementary-school children learn some probability concepts and terms. Each activity requires that children work in groups of 2 or 3 and that opportunities exist for experimentation with the materials. Each one can be presented either as a teacher-directed or learning-center activity. Initially teacher direction is appropriate because you will need to help children learn terms and procedures. Later, learning centers can be used as children's knowledge of the topic and the number and variety of experiments increase. Naturally all center activities should be followed by discussions led by you so that children can refine their understandings and develop appropriate vocabulary.

ACTIVITY 14-4 Rolling Dice

Give each pair of children a pair of dice and a chart.

<table>
<tr><th colspan="11">Tally Sheet</th></tr>
<tr><th>2</th><th>3</th><th>4</th><th>5</th><th>6</th><th>7</th><th>8</th><th>9</th><th>10</th><th>11</th><th>12</th></tr>
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</table>

The children roll the dice and determine the total. Each time the dice are rolled, the total is recorded in the proper column of the chart.

Discuss why certain totals appear more frequently than others on the chart.

Variation: The rolling dice experiment can be simulated with a Logo program. Each roll of each die is determined randomly by Logo.

```
TO DIE
OUTPUT (1 + RANDOM 6)
END
```

Then the 2 dice are added together.

```
TO ROLLDICE
PRINT DIE + DIE
END
```

The student can now roll the simulated dice as many times as necessary to generate information for the chart. Rather than typing ROLLDICE 20 times, use a repeat statement.

```
REPEAT 20 [ROLLDICE]
```

An extended probability experiment with Logo is found in *Nudges: Apple Logo Projects*.[6]

Jerry Shryock describes a sampling activity using colored popsicle sticks.[7]

PROBLEM 14-3 Sampling Popsicle Sticks

Mark 100 popsicle sticks with red dots, 100 with yellow dots, and 200 with blue dots; then place them in a paper bag. Prepare a bag for each group of two or three children.

Tell the children that there are sticks with red, blue, and yellow dots. Challenge them to think of a way to determine whether there are more sticks of 1 color than of the other colors without taking them all from the bag. Through discussion help them see that a random sample process can be used to meet the challenge.

Mix the sticks thoroughly. Have a child draw 1 and note its color. Record the color and return the stick to the bag. Repeat the process until a certain number of draws, say 50, have been completed.

Use the tally of draws to help the children conclude that there are more blue than red or yellow sticks.

There are other aspects of this activity you may want to pursue with your children: Have the children make a graph of the count on the chart. Tell the children that there are 400 sticks; have them tell how many of each color they believe there are. Use the concept of ratio to determine

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[6] Glen Bull, Steve Tipps, and Tim Riordon, *Nudges: Apple Logo Projects* (New York: Holt, Rinehart and Winston, 1985), pp. 233–249.

[7] Jerry Shryock, "Statistical Sampling and Popsicle Sticks," in *Teaching Statistics and Probability* (Reston, Va.: National Council of Teachers of Mathematics, 1981), pp. 45–49.

Tally Sheet		
Color of Dots	Count	Total
Red	𝍸 𝍸 ||||	14
Yellow	𝍸 𝍸 ||	12
Blue	𝍸 𝍸 𝍸 𝍸 ||||	24

the number of each color of stick if there had been 200 red sticks, 400 blue sticks, and so on.[8]

As children mature, their probability activities can become more complex. Problems 14-4 and 14-5 are examples of ones for older children. These activities use beads in a bag. The beads should all be the same size and weight and should feel the same when touched.

PROBLEM 14-4

Drawing Beads

Begin with 2 beads, 1 red and 1 black. Draw a bead from the bag, note its color. Ask "What is the color of the bead that is still in the bag?" Help children conclude that the chance of drawing a certain color of bead is 1 out of 2, or ½. The act of drawing a bead from the bag is called an event; the chance of the event being the drawing of a red bead is 1 out of 2. Have children work in pairs to verify this by drawing a red or a black bead from a bag. Have each pair of children draw and replace a bead 100 times and record their results in a simple table.

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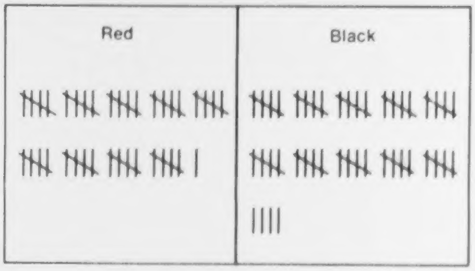
[8] This activity is adapted from the Shryock article.



When each pair has completed 100 drawings, compile a master chart. The master chart should confirm the conclusion that the probability for each event is 1 in 2.

Repeat with other bead combinations:

☐ "What is the probability of drawing a red bead when you have 2 black and 1 red bead in the bag?"

☐ "What is the probability of drawing a black bead when you have 3 red beads and 6 black beads in the bag?"

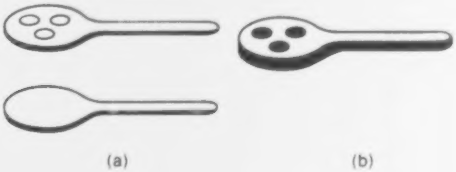
☐ "What is the probability of drawing a black bead when there are 3 red beads and 1 black bead?"

Other readily available materials should be used along with or following work with beads so children can confirm that given events will occur when other types of materials are used. For example, when a red chip and a blue chip are used, children will see that the probability for each event is 1 in 2, just as it is with beads.

Later, children should use larger numbers of beads to discover how to predict probabilities when more than 1 bead at a time is removed from a container. Necessary equipment includes a container with an equal number of red and black beads—perhaps 50 of each—and a device for removing a given number of beads. An easily made device is a spoon fashioned from 2 pieces of Masonite board. The top piece of Masonite contains holes, or bowls, slightly larger than the beads. This piece is glued to the solid piece to complete the spoon (Figure 14-5). Make several spoons, each with a different number of bowls.

FIGURE 14-5

(a) The 2 parts of a probability spoon. (b) The assembled spoon.



PROBLEM 14-5

Probability Spoon

Give each pair of children a bag containing 50 red and 50 black beads and a bead spoon. (A 3-holed spoon is used here. One hole is lettered *A*, 1 is lettered *B*, and 1 is lettered *C*.)

Be sure the children know what events can occur and how to record each one. When there are an equal number of the 2 beads and a 3-holed spoon, 4 events are possible: 3 red, 2 red and 1 black, 1 red and 2 black, and 3 black. These events can be recorded as (3R, 0B), (2R, 1B), (1R, 2B), (0R, 3B) or simply as ordered pairs in which the first numeral refers to the red beads and the second numeral refers to black beads: (3,0), (2,1), (1,2), (0,3). The children should readily see that there is only 1 way to draw 3 red beads or 3 black beads. They will probably need help to see that each of the other 2 events can occur 3 different ways. Use the lettered holes on the spoon to make this clear.

Have children record results as beads are drawn until a chart listing all the possible outcomes is completed.

Events	Bowls		
	A	B	C
(3,0)	R	R	R
(2,1)	R	R	B
	B	R	R
	R	B	R
(1, 2)	B	B	R
	R	B	B
	B	R	B
(0,3)	B	B	B

The chart shows that there are 8 possible outcomes when 3 beads are drawn. Help the children determine the probabilities for each event: (3,0) and (0,3) each have a probability of 1 in 8, whereas (2,1) and (1,2) each have a probability of 3 in 8.

Have other pairs of children draw 2, 4, 5, or 6 beads at a time. (Remember that the greater the number of beads, the greater the amount of time needed to complete the task.)

Comparing tables of events in which different spoons are used will yield an interesting observation concerning the number of possible outcomes when 2, 3, 4, and so on beads are drawn. There are 4 possible outcomes when 2 beads are drawn; the number increases to 8 when 3 beads are drawn; there are 16 possible outcomes when 4 beads are drawn. "Can you see a pattern that will enable us to predict the number of possible outcomes when we draw 5 beads?" (2^5, or 32). "Six beads?" (2^6, or 64.) "Any number of beads?" (2^n.)

A coin-tossing experiment is a practical way to extend children's understanding of outcomes when the number of possible outcomes is increased. A calculator will aid children in computing the different outcomes. When a single coin is tossed, there are only 2 possible ways it can land; the coin can land either heads up or tails up. When 2 coins are tossed, there are 4 different outcomes: H-H, H-T, T-H, T-T. When 3 coins are tossed, these are the possible outcomes: H-H-H, H-H-T, H-T-H, T-H-H, H-T-T, T-H-T, T-T-H, and T-T-T. The number of ways 1, 2, and 3 coins can land are shown in this triangular arrangement of numbers:

	1
1 coin	1—1
2 coins	1—2—1
3 coins	1—3—3—1

List all of the possible outcomes when 4 coins are tossed. What numbers indicate the ways 4 coins can land? Add these numbers to the triangle. Does the triangle give you enough information to predict the number of ways 5 coins can land? If not, list all of the possible outcomes for 5 coins. Extend the triangle to include complete rows for 10 coins.

This arrangement of numbers is called Pascal's triangle, after the 17th-century mathematician, Blaise Pascal, who studied and wrote about it. With it, many interesting questions can be considered:

> What is the sum of each row of the triangle? Can you describe the sequence of numbers that includes each sum?
>
> When 6 coins are tossed, what is the probability of getting 3 heads and 3 tails? What percent of the possible outcomes is this? What is the probability of getting 4 heads and 4 tails when 8 coins are tossed? What percent of the possible outcomes is this? Will the percent of possible outcomes for 5 heads and 5 tails when 10 coins are tossed be greater or less than the percent for 4 heads and 4 tails when 8 coins are tossed? Why do you think this is so?
>
> When you toss 7 coins, what is the probability of getting 3 heads and 4 tails? What is the probability of getting 4 heads and 3 tails? Is each of these probabilities the same percent of the total outcomes? How do you explain this?
>
> When you toss 9 coins, what is the probability that you will get 4 heads? Can you give the probability of getting at least 4 heads?

Other probability activities for children to follow are:

1. Give each child a familiar object—thumbtack, small paper plate, or tetrahedron model with numbers or letters on the faces—and ask each one to determine the probabilities for each object to land a given way after it is tossed or spun. Some of the objects will lead to events for which probabilities can be easily established (for example, the tetrahedron model, assuming it is not weighted, but rather is *fair*) while others will not (paper plate).
2. Have children investigate the probabilities of multiple events. For example, "When we use our 3-bowl spoon, what is the probability of drawing either 3 red or 3 black beads from the container?" "What is the probability that we will draw either all black beads or 2 black beads and 1 red bead from the container?"
3. Ask children to bring games that use dice or spinners for determining moves. Have them determine the probabilities for the occurrence of different events (moves).

Probability Terms

Certain probability terms and concepts can be introduced through discussion of children's activities and hypothetical situations. Terms such as *event, likely event, impossible event, certain event*, and *equally likely event* are the ones that fifth- and sixth-grade children can comprehend. As children engage in experiments, you should capitalize on the action to develop vocabulary. You can tell them that an "event" occurs each time a pair of dice are thrown or beads are drawn from a bag. Events like these can be assigned a number that indicates the probability that a particular event can occur. When 1 coin is tossed, it will come up either heads or tails. The probability of heads or tails coming up is equally likely, so there are 2 "equally likely" events when a coin is tossed. When 3 beads are drawn from a bag containing an equal number of red and black beads, the chance of getting 2 red and 1 black or 2 black and 1 red bead from the bag is more likely than of getting all red or all black beads. Thus getting a combination of colors is a "likely event" in this experiment. It is impossible to get a yellow bead from the bag containing red and black beads, so drawing a yellow bead is an "impossible event." During the bead experiment, the beads that are drawn will be either red or black, so drawing a red or a black bead is a "certain event."

Children's experiments also provide opportunities to develop the concept that each event can be assigned a number that indicates the event's probability of occurring. The likelihood of an event occurring is assigned a number on a scale from 0 to 1. The less likely an event is, the closer to 0 it will be; while the more likely an event is, the closer its number will be to 1.

An impossible event is assigned the number 0, while an event that will occur is assigned the number 1. The probability of Christmas occurring in the middle of summer in the northern hemisphere is 0; while the probability that an airplane will cross the Atlantic Ocean when flying from the United States to Spain in an easterly direction is 1. The probability of getting heads when flipping a coin is 1 in 2, or ½. Another way of stating the probability of getting heads is 50 percent. The probability of getting a 6 when a die is tossed is 1 in 6, which can be stated as ⅙, or approximately 17 percent.

Weather forecasters frequently assign a percent figure when they are indicating the probability of rain or snow in a particular region. These probabilities are not mathematically exact, like ones discussed above, but are based on statistics and knowledge of current conditions that make an "educated guess" possible. Sometimes conditions are such that the probability of rain exists but is low, so the forecaster may assign a figure like 10 percent or 20 percent to indicate that it *might* rain. At another time, when conditions indicate a high probability of rain, the number 80 percent or 90 percent might be assigned. In neither situation is there a certainty that it will rain. An activity during which children track weather forecasts from newspaper or TV reports helps develop the concept of probabilities based on statistics and educated guesses. Choose a time for this activity when there a likelihood of rain or snow.

Summary

Much information is provided the public in tabular and graphic forms, and the ability to use tables and graphs effectively is a valuable problem-solving skill. Children need to develop an understanding of tables and graphs and skill in using them. Children themselves provide the source of most of the information they will organize in tables and graphs. Their birth dates, heights, eye colors, and favorite pizzas, ice-cream flavors, and TV shows are examples of information sources from children. Primary children learn about object, picture, and bar graphs. Older children use picture, bar, line, and circle graphs.

Probability experiments help develop important problem-solving skills as children perform experiments, record and organize data, examine data for patterns, establish numerical odds, and make predictions. They also give knowledge for understanding information from books, TV, and other media sources, and they are fun. Both mathematical and statistical probability should be introduced. Mathematical probability deals with the mathematics of chance, while statistical probability is based on accumulated data pertaining to a particular topic, such as rainfall or temperature. Children's work with probability should be largely investigative; formal instruction should be limited to mature children in upper grades of the elementary school.

Study Questions and Activities

1. It is recommended that children serve as the source of data for their own experiences with tables and graphs. Several types of information about children are named in this chapter. Compile a list that contains at least 10 other topics associated with children that could serve as a basis for table and graph activities. Identify the grade level(s) for which you think each activity is appropriate.
2. Select a topic associated with children that is suitable for a graph experience. Write directions for an activity similar to one from this chapter to illustrate how the topic might be presented to children.
3. A follow-up exercise for Problem 14-3 is having children prepare a graph of the information in the table they made during the popsicle-stick activity. Prepare a suitable graph for the information in the table for Problem 14-3. Explain why you believe your graph is an appropriate one for this activity.
4. Problem 14-4 deals with probabilities for drawing colored beads from a bag. The original activity is followed by 3 questions. Reread the instructions for that activity and study the table for it; then answer the 3 questions.
5. Reread the information about the coin-tossing experiment on page 540, then answer the 4 questions on that page.
6. Design a lesson plan for presenting 1 of the probability activities on page 541 to a group of children.

For Further Reading

Bestgen, Barbara J. "Making and Interpreting Graphs and Tables: Results and Implications from National Assessment." *Arithmetic Teacher*, XXVIII, No. 4 (December 1980), pp. 26–29. The *National Assessment of Educational Progress* test indicates that children can read tables and graphs for single bits of information but have difficulty interpreting graphs and solving problems with information from them. The author offers 5 suggestions for improving children's ability to interpret and use information from graphs.

Burns, Marilyn. "Put Some Probability in Your Classroom." *Arithmetic Teacher*, XXX, No. 7 (March 1983), pp. 21–22. A teacher's curiosity about a statement in the Sunday paper provided the basis for a probability activity for her sixth graders. Her procedures and children's responses are described.

Dickinson, J. Craig. "Gather, Organize, Display: Mathematics for the Information Society." *Arithmetic Teacher*, XXXIV, No. 4 (December 1986), pp. 12–15. Four graphing activities are described. In the first, a teacher helps children select a topic and design a graph. The second uses a computer and *Easy Graph*, a simplified graphing program, for making graphs; while the third and fourth use BASIC or Logo languages to write simple programs for producing graphs.

Fennell, Francis (Skip). "Ya Gotta Play to Win: A Probability and Statistics Unit for the Middle Grades," *Arithmetic Teacher*, XXXI, No. 7 (March 1984), pp. 26–30. A 10-day gaming unit designed to acquaint fifth- to eighth-grade children with games of chance is described. Children are introduced to many probability terms and concepts.

Fraser, Don. "Predicting the Winner of the World Series," *Arithmetic Teacher*, XXXII, No. 2 (October 1984), pp. 34–35. A procedure that uses information about the earned-run averages and number of triples for each team in the 2 major baseball leagues is used to predict the World Series winner. It has been applied to each year since 1969 and was accurate for all but the years 1981 and 1983.

Horak, Virginia M., and Willis J. Horak. "Collecting and Displaying the Data Around Us," *Arithmetic Teacher*, XXX, No. 1 (September 1982), pp. 16–20. Activities based on personal, home, and school grounds data are used with a variety of graphs. Easily made graphs are pictured and described.

———. "Take a Chance," *Arithmetic Teacher*, XXX, No. 9 (May 1983), pp. 8–15. This article deals with mathematics associated with probability. The simplest ones are suitable for primary-grade children, while the others are more appropriate for the intermediate grades.

Johnson, Elizabeth M. "Bar Graphs for First Graders," *Arithmetic Teacher*, XXIX, No. 4 (December 1981), pp. 30–31. A sequence of 7 activities that ends with the construction of a bar graph is described. In addition to the graph itself, children deal with important concepts like "more than," "less than," "as many as," and so on.

Nibbelink, William. "Graphing for Any Grade," *Arithmetic Teacher*, XXX, No. 3 (November 1982), pp. 28–31. Nibbelink describes a way of introducing and making coordinate graphs that is suitable for children in all elementary grades. Step-by-step directions using a variety of data indicate how to use the graph in different grades.

Shaw, Jean M. "Dealing with Data," *Arithmetic Teacher*, XXXI, No. 9 (May 1984), pp. 9–15. Children as young as six can learn about statistical concepts, such as measures of central tendency, measures of dispersion, and correlation, with the activities described in this article. Ways of displaying and interpreting data are discussed.

———. "Making Graphs," *Arithmetic Teacher*, XXXI, No. 5 (January 1984), pp. 7–11. Graphing experiences based on children's lives are described, and ways to help children interpret and make predictions based on their data are suggested.

Woodward, Ernest. "A Second-Grade Probability and Graphing Lesson," *Arithmetic Teacher*, XXX, No. 7 (March 1983), pp. 23–24. Dice and paper grids provide the means for introducing second-grade children to the concept of probability through graphing the outcomes of multiple rolls of the dice.

Yates, Daniel S. "Using Mathematics to Predict," in *Practical Ways to Teach Basic Mathematical Skills*, Monograph (Richmond, Va.: Virginia Teachers of Mathematics, 1978), pp. 148–175. A variety of probability experiments and required materials are described. Most of the activities use easily constructed teacher-made materials.

Appendix A

Sources of Information about Math Anxiety and Equity

Board of Education for the City of Toronto
155 College Street
Toronto, Ontario M5T 1P6 Canada

The Invisible Filter kit is a report and 5 brochures reporting on the possible effects from discontinuing mathematics education early in one's schooling. The kit is designed to inform teachers and parents of the need for mathematics in the complete education of students. The board also distributes a pair of posters: *Dropping Math? Say Good-bye to 82 Jobs* and *Mathophobia Can Cost You A Career*.

Department of Curriculum and Instruction
School of Education
The University of Kansas
Lawrence, KS 66045

COMETS Profile (Career Oriented Modules to Explore Topics in Science) are 24 modules containing more than 100 activities to encourage girls (and boys) to pursue science careers. Each module contains a bibliographical profile of a woman who has pursued a science career.

Math/Science Resource Center
c/o Mills College
Oakland, CA 94615

This is an association of over 1200 scientists, educators, engineers, administrators, business and community persons, and parents who work cooperatively to increase the participation of women in scientific and technical careers. Activities and materials are designed for students of all ages. Network publications that contain material for elementary-school teachers include *Math for Girls and Other Problem Solvers*, *Use Equals to Promote Participation of Women in Mathematics*, *SPACES: Solving Problems of Access to Courses in Engineering and Science*, *Off and Running*, and *Family Math*.

Women's Educational Equity Act Publishing Center
Educational Development Center
55 Chapel Street
Newton, MA 02160

The Women's Educational Equity Act was passed by Congress in 1974 with the goal of promoting educational equity for girls and women in the United States. The U.S. Department of Education provided grants for the development of materials and resources. The catalog *Resources for Educational Equity* contains descriptions of materials and programs for elementary, secondary, and postsecondary levels of education. WEEA Publishing Center maintains a toll-free line (800-225-3088) for persons outside Massachusetts who seek assistance in choosing and implementing programs. The phone number for Massachusetts residents is 1-617-969-7100.

Appendix B

Producers and Distributors of Mathematics Courseware

1. Distributors

Cambridge Development Lab
1696 Massachusetts Ave.
Cambridge, MA 02138

Chambers International
3499 North Federal Hwy., Suite A
Boca Raton, FL 33431

K–12 MicroMedia Publishing
6 Arrow Rd.
Ramsey, NJ 07446

2. Software Producers and Distributors

Collamore Educational Publishing
125 Spring St.
Lexington, MA 02173

CONDUIT
University of Iowa, Oakdale Campus
Iowa City, IA 52242

Control Data Publishing Co.
P.O. Box 261127
San Diego, CA 92126

DLM, Inc.
One DLM Park
Allen, TX 75002

Davidson & Associates, Inc.
3135 Kashiwa St.
Torrance, CA 90505

EduWare Services, Inc.
185 Berry St.
San Francisco, CA 94107

Encyclopaedia Brittanica Educational Corporation
425 North Michigan Ave.
Chicago, IL 60611

Grolier Educational Services
Sherman Turnpike
Danbury, CT 06816

Hartley Courseware, Inc.
133 Bridge
Dimondale, MI 48821

The Learning Company
6493 Kaiser Drive
Fremont, CA 94555

Milliken Publishing Co.
1100 Research Blvd., P.O. Box 21579
St. Louis, MO 63132-0579

Mindscape, Inc.
3444 Dundee Rd.
Northbrook, IL 60062

MECC—Minnesota Educational Computing Consortium
3490 Lexington Ave., North
St. Paul, MN 55126

Scholastic Software
P.O. Box 7501, 2931 McCarty St.
Jefferson City, MO 65102

SVE—Society for Visual Education
1345 Diversey Parkway
Chicago, IL 60614

South-Western Publishing Company
5101 Madison Rd.
Cincinnati, OH 45227

Spinnaker Software Corporation
One Kendall Square
Cambridge, MA 02139

Springboard Software, Inc.
7808 Creekridge Circle
Minneapolis, MN 55435

Sunburst Communication, Inc.
39 Washington Ave.
Pleasantville, NY 10570

Weekly Reader Family Software
Xerox Educational Publications
245 Long Hill Rd.
Middletown, CT 06457

3. Logo

LCSI—Logo Computer Systems, Inc.
9960 Cote Deliesse Rd.
Lechine, PQ H8t 1A1
Canada

Terrapin, Inc. (Logo)
222 Third St.
Cambridge, MA 02142

4. Hardware Manufacturers Selling Software Products

Apple Computer, Inc.
20525 Mariana Ave.
Cupertino, CA 95014

Commodore Business Machines, Inc.
1200 Wilson Dr.
West Chester, PA 19380

IBM
P.O. Box 2150
Atlanta, GA 30055

Radio Shack/Tandy
1500 One Tandy Center
Fort Worth, TX 76102

5. Textbook Publishers with Software Packages

Addison–Wesley Publishing Co.
6 Jacob Way
Reading, MA 01867

D. C. Heath and Company
125 Spring St.
Lexington, MA 02173

Holt, Rinehart and Winston/CBS
383 Madison Ave.
New York, NY 10017

Houghton Mifflin Co./Educational Software Div.
Box 683
Hanover, NH 03755

Random House, Inc., School Division
210 East 50th St.
New York, NY 10022

Scott, Foresman and Company
1900 East Lake Ave.
Glenview, IL 60025

Appendix C

Books for Preschool and Kindergarten Teachers

Baratta-Lorton, Mary. *Mathematics Their Way* (Menlo Park, Cal.: Addison-Wesley, 1976).

———. *Workjobs* (Menlo Park, Cal.: Addison-Wesley, 1972).

———. *Workjobs II* (Menlo Park, Cal.: Addison-Wesley, 1978).

Burton, Grace M. *Towards a Good Beginning: Teaching Early Childhood Mathematics* (Menlo Park, Cal.: Addison-Wesley, 1984).

Camil, Constance. *Young Children Reinvent Arithmetic* (New York: Teachers College Press, 1985).

Copeland, Richard W. *Math Activities for Children* (Columbus, Ohio: Charles E. Merrill, 1979).

Cruikshank, Douglas E., David L. Fitzgerald, and Linda R. Jensen. *Young Children Learning Mathematics* (Boston: Allyn and Bacon, 1980).

Fulton, Oliver. *Mathways for Early Childhood* (New Rochelle, N.Y.: Cuisenaire Co. of America, 1987).

Holt, Michael, and Zoltan Dienes. *Let's Play Math* (New York: Walker and Company, 1973).

Jones, J., and F. Vernon. *Mathtime 1* (New Rochelle, N.Y.: Cuisenaire Co. of America, 1987).

———. *Mathtime 2* (New Rochelle, N.Y.: Cuisenaire Co. of America, 1987).

Nuffield Mathematics Foundation. *Mathematics, the First Three Years* [London: John Murray, 1970]. (The Nuffield Foundation books are published in the United States by John Wiley & Sons, Inc., New York.)

Payne, Joseph N., ed. *Mathematics Learning in Early Childhood* (Reston, Va.: National Council of Teachers of Mathematics, 1975).

Appendix D

Prefixes and Suffixes of Mathematical Terms

Prefixes

bi-, *L.*—twice, two
centi-, *L.*—hundred
deci-, *L.*—ten, tenth
deka-, *Gr.*—ten
dia-, *Gr.*—through
dodeca-, *Gr.*—twelve
geo-, *Gr.*—earth
hecto-, *Gr.*—hundred
hepta-, *Gr.*—seven
hexa-, *Gr.*—six
icosa-, *Gr.*—twenty
isos-, *Gr.*—equal
kilo-, *Gr.*—thousand
milli-, *L.*—thousand
octa-, *Gr.*—eight
para-, *Gr.*—side by side
penta-, *Gr.*—five
peri-, *Gr.*—around
poly-, *Gr.*—many
quadri-, *L.*—four
rect-, *L.*—right
tetra-, *Gr.*—four
tri-, *Gr.* and *L.*—three

Suffixes

-angle, *L.*—angle
-hedron, *Gr.*—side, face
-gon, *Gr.*—angle
-lateral, *L.*—side
-meter, *Gr.*—measure

Note: *L.* indicates Latin; *Gr.* indicates Greek.

Appendix E

Producers and Distributors of Mathematics Learning Aids

Activities Resources
P.O. Box 4875
Hayward, CA 94540

Creative Publications
505 West 110th Street
Oak Lawn, IL 60453

Cuisenaire Company of America, Inc.
12 Church Street
Box D
New Rochelle, NY 10805

Dale Seymour Publications
P.O. Box 10888
Palo Alto, CA 94303

Didax, Inc.
Educational Resources
5 Fourth Street
Peabody, MA 01960

Educat
P.O. Box 2891
Clinton, IA 52735

Nasco
901 Janesville Avenue
Fort Atkinson, WI 53538

or

Nasco West
1524 Princeton Avenue
Modesto, CA 95352

Appendix F

Quick Guide to Activities, Problems, and Games

The following charts list activities, problems, and games found in *Guiding Children's Learning of Mathematics*. Each activity, problem, or game is identified by numerals. The first numeral indicates the chapter in which the activity, problem, or game is found; the second numeral indicates its order within the chapter. The name is given in the second column. Letters in the third column identify the nature of the activity, problem, or game. For example, the letters for Activity 6-2—SR, CL—tell that the activity deals with spatial relationships and classification. (A key for identifying the meanings of letters precedes each chart.) The fourth column indicates whether the activity, problem, or game is used to introduce a new concept (Intro), or provide practice (Prac), enrichment (Enrich), or reinforcement (Rein). The fifth column indicates the grade level(s) for which an activity, problem, or game is recommended. The final column indicates whether the activity, problem, or game is at the concrete (Con), pictorial (Pic), representational (Rep), or abstract (Abs) level of operation.

ACTIVITIES

Prenumber

Key: Classification (CL), Geometry (GEO), Measurement (ME), One-to-One-Matching (OOM), Patterns (PAT), Spatial Relationships (SR)

6-1	Rice-pouring Center	ME	Intro	K-1	Con
6-2	Building-block Center	SR, CL	Intro	K-1	Con
6-3	Structured Materials Center	CL, ME	Intro	K-2	Con
6-4	Dressing Up	CL, ME, OOM	Intro	K-1	Con

6-5	Math Boxes Center	CL, PAT	Intro	K-1	Con
6-6	Matching Canoes and Paddles	OOM	Intro	K-1	Con
6-7	Matching Outlines	OOM	Intro	K-1	Con
6-8	Ordering by Size	OOM, ME, PAT	Intro	K-1	Con
6-9	Sorting by Number of Wheels	CL, OOM	Intro	K-1	Pic
6-10	Comparing Lengths	CL, ME, PAT	Intro	K-1	Con
6-11	Describing Objects	CL, ME, GEO	Intro	K-2	Con
6-12	Listen to the Tom-Tom	PAT	Intro	K-2	Abs
6-13	Long, Short, and In-Between Toots	PAT	Intro	K-2	Abs
6-14	Ordering Geometric Shapes	CL, GEO	Intro	1-2	Con
6-15	Classifying by Color	CL, GEO	Intro	1-3	Con
6-16	Classifying by Color and Shape	CL, GEO	Intro	2-6	Con

Counting, Numerals, and Place Value

Key: Calculator (CA), Classification (CL), Counting (CO), Estimation (EST), Numerals (NUM), Operation (OP), Patterns (PAT), Place Value (PV)

4-1	Calculator Count-Up	CO, CA	Intro	1-3	Abs
6-17	Counting Small Groups	CO	Intro	K-1	Con
6-18	Learning about Zero	CO	Intro	K-1	Con
6-19	The Number Seven	CO, CL	Intro	K-1	Pic
6-20	Arrangements of 7 Objects	CO	Intro	K-1	Con
6-21	Matching Numeral and Set Cards	CO, NUM	Intro	K-1	Rep
6-22	Matching Domino Cards and Numerals	CO, NUM	Intro	K-1	Rep
6-23	Matching Numeral and Set Circles	CO, NUM	Intro	K-1	Rep
6-24	Walking the Number Line I	CO, NUM	Intro	K-1	Rep
6-25	Counting by Touch	CO	Rein	K-2	Con
6-26	Walking the Number Line II	CO, NUM	Rein	K-2	Rep
6-27	Counting On	CO	Rein	K-2	Con
7-1	Trains and Cars	PV, CO, NUM	Intro	1-2	Con

Counting, Numerals, and Place Value *(continued)*

7-2	Place Value with Sticks and Chips	PV, CO, NUM	Intro	1-3	Con
7-3	Place Value with Beans and Sticks	PV, CO, NUM	Intro	1-3	Con
7-4	Pumpkin Seeds	PV, CO, EST	Rein	2-4	Con
7-5	Wipe Out	PV, CA, OP	Rein	3-5	Abs
7-6	Counting by Using a Hundreds Chart	PV, CO, NUM	Rein	1-2	Abs
7-7	Skip-Counting	CO, PAT	Intro	K-2	Con
7-8	Patterns	CO, PAT	Rein	2-4	Abs
7-9	Calculator Counting	CO, PAT, CA	Rein	3-6	Abs

Geometry

Key: Coordinate Plane (CP), Enclosure (ENC), Lines and Points (LP), Plane Figures (PF), Proximity (PROX), Separation (SEP), Order (ORD), Similarity (SIM), Solid Figures (SF), Symmetry (SYM)

8-1	Near and Far	PROX	Int	K-1	Con
8-2	Separation	SEP	Int	K-2	Con
8-3	Balls in a Tube	ORD	Int	K-2	Con
8-4	Enclosure on a Line	ENC	Int	K-2	Con
8-5	Enclosed by a Fence	ENC	Int	K-2	Con
8-6	Enclosed in a Jar	ENC	Int	K-2	Con
8-7	Three-Sided Figures	PF	Int	K-2	Con
8-8	Polygons	PF	Int	2-3	Con
8-9	Open and Closed Curves	PF	Int	2-4	Con
8-10	Four-Sided Figures	PF	Int	3-5	Con
8-11	Segments and Rays	LP	Int	3-5	Rep
8-12	Closed Curves	LP	Int	3-5	Rep
8-13	Planes and Solids	PF, SF	Int	1-2	Con
8-14	Solid Perspectives	SF	Int	4-6	Con
8-15	Similarity	SIM	Int	3-6	Con
8-16	Symmetry	SYM	Int	3-6	Con
8-17	Symmetrical Blobs	SYM	Int	3-6	Pic, Rep
8-18	The Coordinate Plane	CP	Int	3-5	Rep

Operations with Whole Numbers

Key: Addition (ADD), Calculator (CAL), Computer (COM), Counting (CO), Estimation (EST), Division (DIV), Mental Arithmetic (MA), Multiplication (MULT), Number Properties (NP), Place Value (PV), Problem Solving (PS), Subtraction (SUB)

4-2	What Is the Correct Answer?	ADD, SUB, MULT, DIV, CAL	Intro	4-6	Abs
5-1	The Books	SUB, PS	Intro	K-1	Con
5-2	Making up Word Problems	ADD, SUB, PS	Intro	K-2	Abs
5-3	Subtraction Word Problem	SUB, PS	Intro Prac	2-3	Abs
9-1	Introducing Addition	ADD, CO	Intro	K-1	Con
9-2	The Addition Sentence	ADD, CO	Intro	K-1	Con
9-3	Introducing Subtraction	SUB, CO, PS	Intro	K-1	Con
9-4	The Subtraction Sentence	SUB, CO	Intro	K-1	Con
9-5	Adding with Beads	ADD, CO	Rein	K-1	Con
9-6	Adding Objects to a Container	ADD, CO	Rein	K-1	Con
9-7	Subtracting with Blocks	SUB, CO	Rein	K-1	Con
9-8	Dropping Beads	SUB, CO	Rein	K-1	Con
9-9	Cuisenaire Trains	ADD, CO	Intro Rein	K-1	Con
9-10	Addition and Subtraction	ADD, SUB, CO	Intro Rein	K-1	Con
9-11	Addition Sentence Versus Vertical Form	ADD	Intro	1-2	Con Abs
9-12	Addition Chart	ADD	Rein	1	Rep
9-13	Near Doubles	ADD	Rein	1-2	Abs
9-14	Adding One Less Than Ten	ADD	Rein	1-2	Abs
9-15	Brain versus Button	ADD, SUB, CA	Prac	2-5	Abs
9-16	House Numbers	ADD, SUB, CA	Prac	2-3	Abs
9-17	What's My Rule?	ADD, MA, COM	Prac	2-6	Abs
9-18	Commutative Property	NP, ADD	Intro	1-3	Con Abs
9-19	Three Addends	NP, ADD	Intro	1-3	Con Abs
9-20	Higher-decade Addition	ADD, MA	Intro	3-4	Abs
9-21	Adding Tens	ADD, PS	Intro	2	Con
9-22	Regrouping	ADD, PV, PS	Intro	2-3	Con

Operations with Whole Numbers *(continued)*

9-23	Decomposition	SUB, PV, PS	Intro	2-3	Con
9-24	Equal Additions	SUB, MA	Intro, Enrich	4-6	Con Abs
10-1	Buying Gum	MULT, PS	Intro	2-3	Con
10-2	Making Bicycles	MULT, PS	Intro	4-6	Con
10-3	Dividing Beans among Cups	DIV	Intro	1-3	Con
10-4	Dividing Marbles among Bags	DIV, PS	Intro	1-3	Con
10-5	Partitive Situation	DIV, PS	Intro	1-3	Con
10-6	The Multiplication Table	MULT	Rein	3-4	Abs
10-7	Nines Are Easy	MULT, MA	Rein	3-4	Abs
10-8	Products with Nine as One Factor	MULT, MA	Rein	3-4	Abs
10-9	Missing Factors	DIV, MULT, MA	Rein	3-4	Abs
10-10	One Missing Factor	DIV, MULT, MA	Rein	3-4	Abs
10-11	Introducing an Algorithm	MULT, PS	Intro	3	Con
10-12	Regrouping in Multiplication	MULT, PS	Intro	3-4	Con
10-13	Mental Multiplication with Aliquot Parts	MULT, MA	Intro	4-6	Abs
10-14	A Division Algorithm	DIV, PS	Intro	3-4	Con
10-15	Stamp Display	DIV, PS	Rein	3-4	Abs
10-16	Bagging Marbles	DIV, PS	Rein	4-5	Abs
10-17	Estimating Quotients	DIV, EST	Int	4-6	Abs

Fractional Numbers

Key: Common Fraction (CF), Decimal Fraction (DF), Estimation (EST), Number Theory (NT), Numerals (NUM), Operation (OP), Percent (PER), Problem Solving (PS), Tables and Graphs (TG)

11-1	Geometric Regions	CF, PS	Int	K-1	Con
11-2	Berry Pie	CF, PS	Int	1-2	Con
11-3	Congruent Shapes	CF	Int, Rein	3-4	Con
11-4	Fraction Strips	CF, PS	Int, Rein	3-4	Con
11-5	Number Line Charts	CF, PS	Int, Rein	3-4	Rep
11-6	Midpoints	CF, PS	Int	5-6	Rep
11-7	Least Common Multiples	CF	Int	5-6	Abs
11-8	Factorization Process	CF, NT	Int	5-6	Abs
11-9	Simplifying Fractions	CF	Int	5-6	Rep

Fractional Numbers *(continued)*

11-10	Adding Common Fractions	CF, OP	Int	3-4	Con
11-11	Answers Greater Than One	CF, OP	Int	3-4	Con, Rep
11-12	Adding with Unlike Denominators	CF, OP	Int	5-6	Con
11-13	Adding Mixed Numerals	CF, OP	Int	5-6	Con
11-14	Cake Sale	CF, OP	Int	3-4	Con
11-15	Multiplying a Fraction by a Whole Number	CF, OP	Int	4-5	Con
11-16	Multiplying a Whole Number by a Fraction	CF, OP	Int	4-5	Con
11-17	Multiplying with Mixed Numerals	CF, OP	Int	5-6	Con
11-18	Measurement Division with Fractions	CF, OP	Int	5-6	Con
11-19	Finding Reciprocals	CF, OP	Int	5-6	Abs
11-20	Using Reciprocals	CF, OP	Int	5-6	Abs
11-21	Rate Pairs	CF, PS, TG	Int	4-5	Abs
12-1	Tenths	DF	Int	3-4	Con
12-2	Hundredths	DF	Int	3-4	Con
12-3	Decimal Fractions	DF, NUM	Int	3-4	Con
12-4	Adding with Decimal Fractions	DF, OP, PS	Int	3-4	Con, Rep
12-5	Subtracting with Decimal Fractions	DF, OP, PS	Int	4-5	Abs
12-6	Multiplying a Decimal Fraction by a Whole Number	DF, OP, PS	Int	5-6	Rep
12-7	Multiplying a Decimal Fraction by a Decimal Fraction	DF, OP, PS	Int	5-6	Rep
12-8	Multiplying a Whole Number by a Decimal Fraction	DF, OP, PS	Int	5-6	Con
12-9	Partitive Division	DF, OP, PS	Int	5-6	Con
12-10	Measurement Division	DF, OP, PS	Int	5-6	Rep
12-11	Reading Words for Decimal Fractions	DF, NUM, CA	Rein	5-6	Abs
12-12	Placing Decimal Points	DF, OP, EST	Rein	5-6	Abs
12-13	Checking Products	DF, OP, EST	Rein, Prac	5-6	Abs
12-14	Batting Averages	DF, OP, CA	Rein, Prac	5-6	Abs
12-15	Introducing Percent	PER	Int	5-6	Con
12-16	Percent	PER, PS, TG	Int	5-6	Con

Measurement

Key: Area (AREA), Estimation (EST), Linear (LIN), Money (MON), Problem Solving (PS), Temperature (TEMP), Time (TIME), Volume, or Capacity (VOL), Weight, or Mass (WGT)

13-1	Capacity	VOL	Int	K-1	Con
13-2	Heavier and Lighter	WGT	Int	K-2	Con
13-3	Passage of Time	TIME	Int	K-2	Con
13-4	Temperature	TEMP	Int	K-2	Abs
13-5	New Pencil Measurements	LIN	Int	1-2	Con
13-6	Foot Units	LIN	Int	1-2	Con
13-7	Nonstandard Weights	WGT	Int	1-2	Con
13-8	Colored-can Units	VOL	Int	1-2	Con
13-9	Your Step and Meters	LIN	Int	3-4	Con
13-10	How Much is 50 Meters?	LIN, EST	Rein	3-5	Con
13-11	Weighing Produce	WGT	Rein	3-6	Con
13-12	Liters	VOL, EST	Int	3-6	Con
13-13	Finding the Area	AREA	Int	4-6	Con
13-14	Area of a Parallelogram	AREA	Int	5-6	Con
13-15	Cotton-ball Put	LIN	Rein, Prac, Enrich	3-6	Con
13-16	Soda-straw Javelin	LIN	Rein, Prac, Enrich	3-6	Con
13-17	Paper-plate Discus	LIN	Rein, Prac, Enrich	3-6	Con
13-18	Standing Broad Jump	LIN	Rein, Prac, Enrich	3-6	Con
13-19	Standing High Jump	LIN	Rein, Prac, Enrich	3-6	Con
13-20	Build a Clay Bridge	LIN, PS	Rein, Prac, Enrich	3-6	Con
13-21	Build a Soda-straw Bridge	WGT, PS	Rein, Prac, Enrich	5-6	Con
13-22	Sundial	TIME	Int	2-4	Con

Estimation and Mental Arithmetic

5-4	Rounding Off	EST, MA	Intro	4-6	Rep
5-5	Guessing Dots	EST, MA	Intro	2-3	Rep
5-6	Beans in the Jar	EST, MA	Intro	3-6	Con

Miscellaneous

Key: Counting (CO), Integers (INT), Tables and Graphs (TG), Number Theory (NT), Patterns (PAT), Probability (PROB)

7-10	Even and Odd	NT, CO, PAT	Intro	2-4	Con
7-11	Prime and Composite Numbers	NT, CO, PAT	Intro	4-6	Con
7-12	Sieve of Eratosthenes	NT, PAT	Rein	4-6	Abs
7-13	Positive and Negative Numbers	INT, CO	Intro	3-6	Pic
7-14	Vectors	INT	Intro, Rein	4-6	Rep
14-1	The Shoes We Wore Today	TG	Int	K-2	Con
14-2	Pet Statistics	TG	Int	K-2	Con
14-3	Our TV Survey	TG	Int	5-6	Con
14-4	Rolling Dice	PROB, TG	Int	3-4	Con

PROBLEMS

Key: Account for All Possibilities (AAP), Act it Out (AO), Break Problem into Smaller Parts (BP), Break Set (BS), Guess and Check (GC), Identify Wanted-Given Information (IWG), Use a Model (MOD), Look for a Pattern (PAT), Use Picture or Diagram (PD), Restate the Problem (RP), Use Table and/or Graph (TG), Write Mathematical Sentence (WS)

5-1	T-Shirt Patterns	PAT	Intro	K-3	Con
5-2	Arithmetic Patterns	PAT, CA	Intro	2-5	Abs
5-3	Triangular Numbers	PAT, CA	Rein	5-6	Con
5-4	The Towers of Hanoi	PAT, CA	Rein, Enrich	5-6	Con
5-5	A Mystery Sequence	PAT	Enrich	5-6	Abs
5-6	How Many Birds?	PD	Intro	2-3	Pic
5-7	Leaving Adamsville	PD	Intro	3-5	Rep

Problems *(continued)*

5-8	The Dog Kennel	PD	Intro	3-5	Rep
5-9	The Math Teacher	PD	Rein	5-6	Rep
5-10	A Line of Robots	MOD	Intro	3-5	Con
5-11	Cycle Riders	MOD	Intro, Rein	3-6	Con
5-12	Stamp Collectors	MOD	Intro, Rein	5-6	Con
5-13	Our Favorite Ice Cream Flavors	TG	Intro	1-3	Pic, Rep
5-14	Christmas Ornaments	TG	Intro	1-3	Pic, Rep
5-15	Two Unique Rectangles	TG	Intro	5-6	Pic, Abs
5-16	Buying Apples	GC	Intro	4-6	Abs
5-17	The Ice Cream Parlor	AAP, TG	Intro	3-5	Pic, Rep
5-18	Buying Fertilizer	AAP, TG	Intro, Rein	4–6	Pic, Rep, Abs
5-19	Hand Shakes	AO	Intro	3-6	Con
5-20	Horse Trading	AO	Intro	4-6	Con
5-21	Airplane Trip	WS, PD	Intro	2-4	Abs, Rep
5-22	Buying Groceries	WS, MON	Intro, Rein	3-5	Abs
5-23	Measuring Shadows	BP, CA, ME, GEO	Intro	4-6	Con
5-24	At the Five and Dime	RP, MON	Intro	4-6	Abs
5-25	Wine and Cork	RP, MON	Intro	4-6	Abs
5-26	The Farmer's Trees	BS, PD	Intro	4-6	Rep
5-27	Checkers	BS	Intro	3-6	Abs
5-28	Triangles	BS, MOD	Intro	4-6	Con
5-29	Sixty-three Dollars	BS, MOD	Intro, Rein	4-6	Rep Abs
5-30	Candlestick Park	BP, IWG	Rein	3-6	Abs
5-31	Bob's Marbles	BP, IWG, MOD	Rein	3-6	Con, Abs
5-32	Estimation	OP, EST, MA	Intro	3-6	Abs
5-33	More Estimation	OP, EST, MA	Intro	3-6	Abs
7-1	Prime and Composite Numbers	NT, CAL	Rein, Enrich	5-6	Abs
7-2	Testing Goldbach's Conjecture	NT	Enrich	5-6	Abs
7-3	Studying Prime and Composite Numbers	NT	Rein, Enrich	5-6	Abs

Problems *(continued)*

8-1	Diagonals	TG	Rein	5-6	Con
8-2	Nets of Cubes	GC, MOD, DP	Rein	5-6	Con, Pic
8-3	Lines of Symmetry	TG, AAP, GC	Rein	5-6	Con
8-4	Rotations	TG, AAP, GC	Rein	5-6	Con
8-5	Rotational Symmetry	TG, AAP, GC	Rein	5-6	Con
14-1	Favorite TV Shows at Our School	TG	Rein	3-6	Pic
14-2	Making Christmas Ornaments	TG	Int	5-6	Rep
14-3	Sampling Popsicle Sticks	TG, PROB	Rein	4-6	Con
14-4	Drawing Beads	TG, PROB	Rein	4-6	Con
14-5	Probability Spoon	TG, PROB	Rein	5-6	Con

GAMES

Key: Calculator (CA), Counting (CO), Integers (INT), Mental Arithmetic (MA), Operation (OP), Place Value (PV)

7-1	Beans and Sticks	PV, CO, NUM	Rein	2-4	Con
7-2	Spin to Win	PV, NUM, MA	Rein	3-6	Abs
7-3	Expanded Notation Concentration	PV, NUM, MA	Rein	2-3	Abs
7-4	Banker's Game	PV, CO	Rein	3-6	Con
7-5	Cricket Jumps	INT	Rein, Enrich	3-6	Rep
7-6	Checks and Bills	INT, MA	Rein, Enrich	4-6	Abs
9-1	Addition Bingo	OP	Prac	2-5	Abs
9-2	Roll and Add	OP, PV, MA, CA	Prac	3-6	Abs
9-3	Bull's-eye	OP, MA, CA	Prac, Enrich	3-6	Abs
9-4	NIM	OP, MA, CA	Prac, Enrich	3-6	Abs

Note: Many games can be adapted for topics other than the one for which they are used in this book. You should consider other topics or levels of difficulty to adapt these games for your children.

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